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THE

ELEMENTS OF ALGEBRA

DESIGNED FOR

THE USE OF SCHOOLS,

BY

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ERRATA.

Page 17, line 5, for $\sqrt{b}$ read $\sqrt[q]{b}$.
" 119, line 5, for , read $\times$.

Answers to the Examples.

9, 11, for $\dfrac{1+\sqrt{2}}{\sqrt[3]{2}}$ read $2+\sqrt{2}$.
24, 8, for $\bar{1}.4982833$ read $\bar{1}.8316167$.
44, 4, for m^3 in two places read m^2.

Answers to Miscellaneous Examples.

215, for 144 : 235 read 432 : 625.

Answers to Equation Papers.

18, 4, for $\sqrt{(2a-1)}$ read $\sqrt[3]{(2a-1)}$.

ALGEBRA.

Part II.

CHAPTER I.

ELEMENTARY RULES.

*** *The bracket* ( ) *will be used to refer to Articles in Part* I., [ ] *to Articles in Part* II.

1. (1) 'ALGEBRA is the European corruption of the first words of an Arabic phrase, which may be thus written, *al jebr e al mokabalah*, meaning *restoration and reduction*. The earliest work on the subject is that of Diophantus, a Greek of Alexandria, who lived between A.D. 100 and A.D. 400; but when cannot be well settled, nor whether he invented the science himself, or borrowed it from some Eastern work. It was brought among the Mahommedans by *Mohammed ben Musa*, between A.D. 800 and A.D. 850, and was certainly derived by him from the Hindoos. The earliest work which has yet been found among the latter nation is called the *Vija Ganita*, written in the Sanscrit language, about A.D. 1150. It was introduced into Italy from the Arabic work of Mohammed, about A.D. 1200, by *Leonardo Bonacci*, called Leonard of Pisa; and into England by a physician, named *Robert Recorde*, in a book called *The Whetstone of Witte*, published in the reign of Queen Mary, 1557.'—*De Morgan*.

2. (5) More generally, the terms *positive* and *negative*, with the corresponding signs, may be used of any quantities, which are so opposed to each other in character, that any number of units of the one taken together with the same number of units of the other, would neutralize each other.

Thus, as in (15), the *property* and *debts* of a person may be treated as positive and negative quantities; and so also a motion of a feet in one direction may be called a positive motion, and

in the opposite direction, a negative one. In such cases, it matters not which of the two quantities we call the positive; the opposite, or neutralizing one, will be the negative.

3. It will be plain that all the algebraical rules deduced for the Addition and Subtraction of positive and negative quantities, according to their original definition, will apply also to such as these; because, in arriving at these rules, no reasoning has been employed, except what depends upon the property they all share in common, viz. of neutralizing each other, when taken together in equal amount. It will only be necessary in the final results in each case, to give the proper interpretation to the signs + and –, which may occur in it, in accordance with what we have taken them to represent.

Thus (14. Ex. 1) suppose a man has *gained* sums of £$3a$, £$5a$, and £$6a$, and *lost* sums of £$5a$ and £$2a$: if we choose to use + to represent the character of the former, then – will represent that of the latter, and his means will be expressed in £'s by $+3a$, $+5a$, $+6a$, $-5a$, $-2a$, the sum of which is $+7a$, that is, he will have *gained* £$7a$, over and above what he has lost: or if we choose to use – for the former, then + must be used for the latter, and the result would be $-7a$, which must have the same interpretation as the former result, because we have now taken the sign – to represent the sums *gained*. Again, if we suppose $3a$, $5a$, $6a$ to represent spaces, (numbers of feet for instance,) which a man has travelled in one direction, then $-5a$, $-2a$, will represent similar spaces, travelled by him in the opposite direction; and if these be now *taken together* in such a manner as to form one *sum*, (that is, if we suppose them to be travelled over by the man *consecutively*, so that he begins to describe the second, in its proper direction, from the point where he ends the first, &c.) then his motion will upon the whole be equivalent to one of $+7a$ from his starting place, that is, of $7a$ feet in the first direction.

4. So also the Rule of Signs in Subtraction applies to all such quantities, viz. that to *subtract* a number of *positive* units is the same as to *add* the same number of *negative* units, and *vice versâ*.

For since any given quantity is not altered by adding to it a positive units and a negative units, let us suppose this done, and then take away, or *subtract*, the a *positive* units; we shall then

have remaining the original given quantity with the *a negative* units *added* to it: and, in like manner, if we *subtract* the *a negative* units, we shall have remaining the original quantity with the *a positive* units *added* to it.

It follows then that to subtract any quantity, positive or negative, we have only to change its affection and add it.

5. It will now be sufficiently apparent that the signs + and − are really used for two distinct purposes, either, *Arithmetically*, as signs of *operation*, to connect quantities with one another by Addition or Subtraction, or, *Algebraically*, as signs of *affection*, to denote the qualities of the quantities before which they stand.

It is, strictly speaking, only in this latter sense that they are found before quantities standing alone, unconnected with others. Thus (5), even in the case of abstract numbers, $+a$ would denote a number *to be added*, and $-a$ a number *to be subtracted*, where the epithets, *to be added* or *subtracted*, express the *qualities* of the numbers, as additive or subtractive quantities, whether or not it may be necessary to connect them with others; just in the same way as $-£a$ might represent a *loss*, whether or not we take it into consideration with other gains or losses.

If, however, it should be required to connect together by Addition or Subtraction any of these quantities, each with its own proper affection, the signs + and − might be used for this purpose also, as signs of operation. Thus the *sum* of the quantities in [2] is strictly $(+3a)+(+5a)+(+6a)+(-5a)+(-2a)$; and so if we wished to *subtract* the latter three from the sum of the others, the result would be expressed by $(+3a)+(+5a)-(+6a)-(-5a)-(-2a)$. But here the reasoning of [4] applies, and teaches us that we may change the − sign of operation into a + sign, by changing also the sign of affection of the quantity before which it stands. By this means the latter result would become $+(+3a)+(+5a)+(-6a)$ $+(+5a)+(+2a)$; and now in both results the signs of operation, being all +, may be omitted and understood to be implied with the signs of affection, so that the two results become

$$+3a+5a+6a-5a-2a=+7a, \quad +3a+5a-6a+5a+2a=+9a;$$

where each sign must be understood to express both the *affection* of the quantity before which it stands, and also the *operation*, viz. Addition, by which it is connected with the others.

That this is really the case will be made more evident by observing that, if the quantities represented as in [3] *gains* and *losses*, we might read the first of the above results thus—*a gain of* $3a$ and *a gain of* $5a$ and *a gain of* $6a$ and *a loss of* $5a$ and *a loss of* $2a$ amount to *a gain of* $7a$—where the words *gain of* and *loss of* express the force of the + and – as signs of affection, and the word *and*, which connects the quantities together, expresses the + sign of operation, understood to be implied with each sign of affection.

6. From the foregoing considerations, the rule of signs in Multiplication, which in (19) was rather inferred than proved by the method there adopted, may be thus more distinctly exhibited.

In all cases of Multiplication of two quantities, one of them, suppose the former, must be a mere abstract number, expressing the number of times (integral or fractional) the other is repeated; while the latter may be a number of abstract units also, or a number of concrete units, as *pounds*, *feet*, &c., and, in either case, may be of some affection, *positive* or *negative*, where these words will of course have the particular meanings appropriate to each case. The sign then of the latter quantity will be a sign of *affection;* but that of the former can only be a sign of *operation.*

Thus $+b$ denotes b *positive* units, and $a \times (+b)$ denotes a times b positive units, or ab positive units, which, of course, is expressed by $(+ab)$, nothing, however, being yet said as to how these ab positive units are to be connected with others. But $+a \times (+b)$ implies that they are to be *added*, and this, as in [5], would be written simply with the sign of affection $+ab$, the + sign of operation being understood with it; whereas $-a \times (+b)$ implies that the ab *positive* units are to be *subtracted*, which [4] is the same as saying that ab *negative* units are to be *added*, and therefore this product is expressed by $-ab$, the + sign of operation being understood as before. In like manner it follows that $+a \times (-b)$, which implies that ab *negative* units are to be *added*, is expressed by $-ab$, and $-a \times (-b)$, which implies that ab *negative* units are to be *subtracted*, that is, by [4], ab *positive* units are to be *added*, is expressed by $+ab$, the + sign of operation being understood in each case to be implied with the sign of affection.

If any number of algebraical quantities are to be multiplied together, such as $+a$, $-b$, $+c$, $-d$, all except one must represent

mere abstract numbers, expressing the numbers of times of repetition in forming the product. The signs of these are therefore signs of operation only, upon which, however, will depend whether the other given quantity retains or not its original affection: thus, by the previous reasoning, $+c\times(-d)=-cd$, $-b\times(-cd)=+bcd$, $+a\times(+bcd)=+abcd$.

7. (28) It is often convenient to employ only the coefficients in multiplication and division. A single example will sufficiently explain the method to the intelligent student.

Thus for (28 Ex. 2) we may proceed as follows:

$$\begin{array}{r|l}
3-4) & 6-17+\ 0+16\ (2-3+4 \\
& \underline{6-\ 8} \\
& -\ 9+\ 0 \\
& \underline{-\ 9+12} \\
& \qquad -12+16 \\
& \qquad \underline{-12+16}
\end{array}$$

Here we insert the term 0 in the dividend, because there is no term in it with xy^2. If we had not done this, we should have been taking down $+16$ (the coeff. of y^3) over $+12$ (the coeff. of xy^2). This is the only point requiring attention in using this method, *to insert a cypher wherever a term is wanting.*

Let the student apply this to (Ex. 12. 9–12).

8. Another still shorter method of division is by what is called Horner's *synthetic* method, as follows.

Ex. 1. Divide $6x^5+5x^4-17x^3-6x^2+10x-2$ by $2x^2+3x-1$.

Let us denote the dividend in this case by D and the quotient by Q: then $D=Q(2x^2+3x-1)$, or $2x^2Q=D+Q(-3x+1)$ by means of which result the quotient Q may be obtained as below.

$$\begin{array}{r|l}
2x^2 & 6x^5+5x^4-17x^3-\ 6x^2+10x-2 \\
-3x & \qquad\ -9x^4+\ 6x^3+12x^2-\ 6x \\
\underline{+1} & \qquad\qquad\ \ \underline{+\ 3x^3-\ 2x^2-\ 4x+2} \\
& 3x^3-2x^2-\ 4x+\ 2
\end{array}$$

In the left-hand vertical column we set the divisor, with the signs changed of all its terms except the first. Now by reference to the result before obtained, since the dividend is of *five* dimensions and the divisor of *two*, Q can only be of *three* dimensions, and $3xQ$ only of *four;* and therefore, since $2x^2Q=D+Q(-3x+1)$, the *first* term of Q would be obtained by dividing the first term of D, which is the only one of *five* dimensions, by $2x^2$: this gives $3x^3$, which we place below, and then multiply $-3x$ and $+1$ by it,

placing the results $-9x^4$ and $+3x^3$ diagonally, as above. Then, by similar reasoning, it is plain that the only part of $D-3xQ+Q$ which can be of *four* dimensions is the sum of the terms $5x^4$ and $-9x^4$, that is, $-4x^4$, dividing which by $2x^2$ we get $-2x^2$, the *second* term in the quotient. Multiplying this again by $-3x$ and $+1$, and placing the results, $6x^3$ and $-2x^2$, diagonally, as before, it follows that the only part of $D-3xQ+Q$ which can be of *three* dimensions is the sum of the terms $-17x^3$, $+6x^3$, $+3x^3$, that is, $-8x^3$, dividing which by $2x^2$, we get $-4x$, the *third* term in the quotient, and so on.

If we omit the coefficients the sum may be worked as follows:

$$\begin{array}{r|l}
2 & 6+5-17-\ 6+10-2 \\
-3 & \quad -9+\ 6+12-\ 6 \\
+1 & \qquad +\ 3-\ 2-\ 4+2 \\
\hline
 & 3-2-\ 4+\ 2
\end{array}
\qquad \textit{Ans. } 3x^3-2x^2-4x+2.$$

When the coefficient of the first term of this divisor is *unity*, no division will be required, and this method will be found very easy of application, as in the following Example, where also a remainder occurs.

Ex. 2. Divide $5x^4-4x^3+3x^2-2x+1$ by x^2-3x+5.

$$\begin{array}{r|r|l}
1 & 5-\ 4+\ 3 & -\ 2+\ 1 \\
+3 & +15+33 & +33 \\
-5 & -25 & -55-55 \\
\hline
 & 5+11+11 & -24-54
\end{array}
\qquad \textit{Ans. } 5x^2+11x+11-\frac{24x+54}{x^2-3x+5}.$$

It is advisable to draw a second vertical line as above, marking the point at which the remainder begins to be formed, viz. with as many vertical columns to the right of it as are less by unity than the number of terms of the divisor.

Let the student apply this method also to (Ex. 12. 9–12).

9. The result of (Ex. 13. 12), in which it is seen that the remainder is of exactly the same form as the dividend, with a in the place of x, is only a particular case of the following general one.

Let $f(x)$ denote any *function* of x, by which is meant any quantity whatever involving x; and suppose $f(x)$ divided by $x-a$, as far as possible, so that the remainder R may no longer contain x; then if Q be the quotient, we have $\dfrac{f(x)}{x-a}=Q+\dfrac{R}{x-a}$, or $f(x)=Q(x-a)+R$.

Now since R no longer contains x, it will remain the same, whatever value x may have: let us then give to x the value a; then $x-a=0$, and $\therefore f(a)=R$, or R is a quantity of precisely the same form as $f(x)$, with a in the place of x.

So also, if $f(x)=Q(x+a)+R$, by putting $x=-a$, we have R, (that is, the remainder when $f(x)$ is divided by $x+a$) $=f(-a)$.

Thus $(x^2+7x-3)\div(x-2)$ gives a remainder $2^2+7.2-3=15$,

$(ax^2+bxy+cy^2)\div(x+m)$ gives a remainder $am^2-bmy+cy^2$;

whereas $(3x^2+x-10)\div(x+2)$ gives a remainder $12-2-10=0$, and therefore $3x^2+x-10$ is exactly divisible by $x+2$.

10. It may be noticed that the law of formation of the quotients in (30), in the division of $a^n \pm x^n$ by $a \pm x$, holds also in those cases in which the division cannot be *exactly* performed; but here, as in (28. Ex. 5, 6), there will be always a remainder $+2x^n$ or $-2x^n$, according as the last term of the dividend is $+x^n$ or $-x^n$.

$$\text{Thus } \frac{a^3-x^3}{a+x}=a^2-ax+x^2-\frac{2x^3}{a+x},\quad \frac{a^4+x^4}{a-x}=a^3+a^2x+ax^2+x^3+\frac{2x^4}{a-x}.$$

11. The above results and those of (30) are easily deducible from [9].

For let $a^n+x^n=Q(a+x)+R$, a being here taken as the letter of reference, so that R no longer contains a: for a write $-x$; then, if n be odd, $a^n=-x^n$, and $0=0+R$, or $R=0$; but, if n be even, $a^n=x^n$, and $2x^n=0+R$, or $R=2x^n$: and so in the other cases.

It follows also from (30) that x^m-x^n or $x^n\ (x^{m-n}-1)$ is always div. by $x-1$, and also by $x+1$ if $m-n$ be even, that is, m and n both even or both odd; and so x^m+x^n is div. by $x+1$, if $m-n$ be odd.

Ex. 1.

1. Write down the remr in the divn of $(x-1)^3$ by $x\pm 2$, and $x+3y$.
2. Shew without divn that $x^2+2x-15$ is div. by $x-3$ and $x+5$, and $2x^3+3x^2-98x-147$ by $x\pm 7$.
3. Shew without divn that $x^3-2ax^2+(a^2+b)x-ab$ is div. by $x-a$, and $x^3+(2a+b)x^2+(a^2+2ab)x+a^2b$ by $x+a$.
4. Write down the quotients of a^6+x^6 by $a-x$, $25x^2+1$ by $5x-1$, $4x^2+9$ by $2x+3$, $16m^4+n^4$ by $4m^2-n^2$.
5. $9m^2n^2+4$ by $3mn+2$, $1+8x^3$ by $1-2x$, $27x^3+1$ by $3x-1$.
6. $1+16x^4$ by $1-2x$, a^4b^4+16 by $ab-2$, a^5-32b^5 by $a+2b$.
7. $(a+b)^2+4c^2$ by $a+b+2c$, $(x+y)^3+z^3$ by $x+y-z$.
8. $(x+y)^3-8z^3$ by $x+y+2z$, $8(x+y)^3-z^3$ by $2x+2y+z$.

12. If in (43) we use the symbol Σ to denote *the sum of all such terms as*, we may write the expression (i) in an abridged form,

$$(a+b+c+\&c.)^2 = \Sigma(a^2) + 2\Sigma(ab),$$

where $\Sigma(a^2) = a^2 + b^2 + c^2 + \&c.$, $\Sigma(ab) = ab + ac + bc + \&c.$

13. A similar expression may be obtained for the *cube* of any multinomial, as follows.

$$(a+b+c+\&c.)^3 = a^3 + 3a^2(b+c+\&c.) + 3a(b^2+2bc+\&c.) + \&c.,$$

where we have three distinct forms or *types* of terms, viz. a^3, $3a^2b$, and $6abc$; and it is manifest, from the principle of Symmetry, that the whole expression will consist of all possible modifications of these three forms, putting b, c, &c. in the place of a: hence

$$(a+b+c+\&c.)^3 = \Sigma(a^3) + 3\Sigma(a^2b) + 6\Sigma(abc),$$

where $\Sigma(a^2b) = a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 + \&c.,$

$\Sigma(abc) = abc + abd + acd + bcd + \&c.$

14. Another form of the above result may be thus obtained:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + (3a^2 + 3ab + b^2)\,b = a'^3, \text{ suppose,}$$

$$(a+b+c)^3 = (a'+c)^3 = a'^3 + (3a'^2 + 3a'c + c^2)\,c$$

$$= a^3 + (3a^2 + 3ab + b^2)\,b + (3a'^2 + 3a'c + c^2)\,c = a''^3;$$

$$(a+b+c+d)^3 = (a''+d)^3 = a''^3 + (3a''^2 + 3a''d + d^2)\,d = \&c.$$

Comparing this result with (54), we see more completely that the sum of all the subtrahends in the operations for the extraction of the cube root $=(a+b+c+d+\&c.)^3$.

15. The methods of (47, 54) may of course be applied to extract the roots of *surd* algebraical quantities to any number of terms: but these may generally be found more easily by the Bin. Theor.

Ex. 2.

Find the square and cube roots, each to four terms, of

1. $1+x$. 2. a^2-b. 3. $2a^2$ or a^2+a^2. 4. $1-2x+3x^2-4x^3+\&c.$

---

16. In the extraction of the *square* root, when $n+1$ figures are found in the root, n more may be found by merely dividing the last remainder by the last trial-divisor.

For let N be the number whose square root is to be found, a the part already found in the root, (consisting of the $n+1$ figures with n cyphers after them, that is, altogether, of $2n+1$ figures,)

and x the remaining part of the root, consisting of n figures; then

$$N = a^2 + 2ax + x^2, \text{ and } \frac{N - a^2}{2a} = x + \frac{x^2}{2a}.$$

Now $N - a^2$ is the remainder at this stage of the operation, and $2a$ the trial-divisor; if then we can shew that $\frac{x^2}{2a}$ is a *proper fraction*, it will follow that the *integer* obtained by dividing $N - a^2$ by $2a$ will be x, the remaining part of the root.

But since x contains n figures, $x < 10^n$, which has $n + 1$ figures, and $x^2 < 10^{2n}$; and since a contains $2n + 1$ figures, $a > 10^{2n}$; hence $\frac{x^2}{2a} < \frac{10^{2n}}{2.10^{2n}} < \frac{1}{2}$, and is therefore a proper fraction.

Ex. Find the square root of 5489031744.

Here there will be *five* figures in the root, as we see by pointing: if then we find *three* of these by the usual method, we may obtain the other *two* by division.

```
        5489031744 (740
        49
        --
   144) 589
        576
        ---
148000) 13031744 (88
         1184000
         -------
         1191744
         1184000
         -------
            7744
```

```
     5489031744 (74088
     49
     --
144) 589
     576
     ---
1480) 13031
      11840
      -----
       11917
       11840
       -----
        7744
```

In the first form of the above example, two cyphers are added to the trial-divisor 1480, because the divisor $2a$ is not really 2×740 but 2×74000: but these may be omitted in practice as in the second form, and then we may take down figure by figure in performing the division. It may be noticed that the complete quotient is $88 + \frac{7744}{148000}$, where $7744 = 88^2 = x^2$, as it should be.

17. In the extraction of the *cube* root when $n + 2$ figures are found in the root, n more may be found by division.

For, as in [16], $N = a^3 + 3a^2x + 3ax^2 + x^3$, and $\frac{N - a^3}{3a^2} = x + \frac{x^2}{a} + \frac{x^3}{3a^2}$: and here $x < 10^n$, and a, since it now contains $2n + 2$ figures, is $> 10^{2n+1}$; $\therefore \frac{x^2}{a} + \frac{x^3}{3a^2} < \frac{10^{2n}}{10^{2n+1}} + \frac{10^{3n}}{3.10^{4n+2}} < \frac{1}{10} + \frac{1}{3.10^{n+2}} < 1.$

Ex. Find the cube root of 5489031744.

Here there will be *four* figures in the cube root, and of these we must find *three* by the rule, and then may obtain *one* more by division. We may omit the cyphers as in [16 Ex.], and so take down, as before, only one figure at a time in the division.

		5̇489̇031̇744̇ (1764
		1
37	300	4489
	259	
	559	3913
516	86700	576031
	3096	
	89796	538776
	92928)	372557
		371712
		84544

Here also we see that the complete quotient is $4+\frac{84544}{9292800}$, where $84544 = 3 \times 1760 \times 4^2 + 4^3 = 3ax^2 + x^3$, as it should be.

18. In applying the above methods to approximate to the square or cube root of a *surd*, it must be noticed that there *may* be an error of *unity* made in the number found by the divⁿ.

For suppose $a+x$ to be, as in [16], the approximate square root of the given surd N: then $N > (a+x)^2$ but $< (a+x+1)^2$, and, $\therefore \frac{N-a^2}{2a}$ lies between $x+\frac{x^2}{2a}$ and $x+1+\frac{(x+1)^2}{2a}$: whence it appears that the integral quotient of $N-a^2$ by $2a$ *may* be $x+1$ instead of x.

And the same may be shewn in the case of the cube root.

19. Similar methods to those in (55) may be applied to extract the 4th, 5th, &c. roots of numbers: thus for the 4th root we should point every *fourth* figure, beginning from the last, and find the number whose 4th power is next less than the first period; calling this a, we should then form the trial-divr $4a^3$, and finding b by dividing the first period by it, we should then form the quantity $4a^3b+6a^2b^2+4ab^3+b^4$, and subtract it; then taking down the next period, and forming the next trial-divisor $4(a+b)^3=4a'^3$, we find c, and subtract $4a'^3c+6a'^2c^2+4a'c^3+c^4$; and so on.

CHAPTER II.

GREATEST COMMON MEASURE, FRACTIONS, SURDS, &C.

20. The following is the Proof usually given of the method of finding the G. C. M. of two algebraical quantities.

Let $a = \alpha a'$, $b = \beta b'$, be the two given quantities, where α, β, represent the product of any simple factors they contain. Then if α, β, be prime to each other, the G. C. M. of a and b must be the same as that of a' and b': but if α, β have any common factor, this must be set aside, and the product of it and the G. C. M. of a' and b' will be the G. C. M. of a and b. Hence we may confine our attention to finding the G. C. M. of two quantities, a' and b', which contain no simple factors.

Suppose a' to be not of lower dimensions than b'. Then if the first term of a' be not exactly divisible by that of b', multiply a' by some simple factor α', which shall make it so divisible; and now divide $\alpha'a'$ by b', with quotient p and remainder c.

Again, let $c = \gamma c'$, where γ represents the product of all the simple factors in c; and if the first term of b' be not exactly divisible by that of c', multiply b' by some simple factor β', which shall make it so divisible; and now divide $\beta'b'$ by c' with quotient q and remainder d.

In like manner, divide $\gamma'c'$ by d' with quotient r, and no remainder: then d' shall be the G. C. M. of a' and b'.

$$\text{For } \alpha'a' - pb' = c = \gamma c', \quad \beta'b' - qc' = d = \delta d', \quad \gamma'c' - rd' = 0:$$

and (i) by (63) every C. M. of a' and b' measures $\alpha'a' - pb'$ or c, and $\therefore c'$, (since γ can have no factor common to a' and b', which contain no simple factor,) and $\therefore \beta'b' - qc'$ or d, and $\therefore d'$;

also (ii) d' measures $\gamma'c'$, and $\therefore c'$ (since γ' can have no factor in common with d' which has no simple factor,) and $\therefore qc' + \delta d'$ or $\beta'b'$,

$$\text{and } \therefore b', \text{ and } \therefore pb' + \gamma c' \text{ or } \alpha'a', \text{ and } \therefore a':$$

so that (i) every C. M. of a' and b' is a M. of d', and (ii) d' is a C. M. of a' and b'; $\therefore d'$ is the G. C. M. of a' and b'.

21. (60. Ex.) As a further illustration of the Remark on this Example, it may be observed that it would be very unsafe to infer

the G. C. M. of two *arithmetical* numbers, obtained by giving some numerical value to the letters of two *algebraical* quantities, from the G. C. M. of the two latter.

Thus the G. C. M. of $3x^2+x-2$ and $3x^2+7x-6$ is $3x-2$: but if we put $x=3$, these quantities become 28 and 42, whose G. C. M. is 14, whereas the num. value of $3x-2$ would be 7. The fact is that $3x^2+x-2=(3x-2)(x+1)$, and $3x^2+7x-6=(3x-2)(x+3)$; and here, beside the common factor $3x-2$, the two factors $x+1$, $x+3$, which algebraically have no common factor, will have the common factor 2, whenever we give x the value of any odd number.

22. The G. C. M. of *three* quantities, a, b, c, may be obtained, by finding first the G. C. M. (x suppose) of a and b, and then the G. C. M. of x and c: and similarly for *four* or more quantities.

Or we may obtain the G. C. M. of a, b, c, d, by finding x the G. C. M. of a and b, and y the G. C. M. of c and d: then the G. C. M. of x and y will be that required: and so in other cases.

23. The following are instances of *literal* coefficients.

Ex. 2*.

Find the G. C. M. of

1. $x^4-px^3+px^2-p^2x$ and px^2-p^3.
2. $x^3+(a+1)x^2+(a+1)x+a$ and $x^3+(a-1)x^2-(a-1)x+a$.
3. $px^3-(p-q)x^2+(p-q)x+q$ and $px^3-(p+q)x^2+(p+q)x-q$.
4. $x^3+x^2y^2+xy+y^3$ and $x^4+yx^3-2y^2x^2-2y^3$.
5. ax^3-bx^2+ax-b and $ax^3+(a-b)x^2+(a-b)x-b$.
6. $ax^3-(a-b)x^2-(b-c)x-c$ and $2ax^3+(a+2b)x^2+(b+2c)x+c$.
7. $p^2x^3+p(1-p^3)x^2+(1-p^2)x-p$ and x^3-p^3.
8. $ax^3-(a^2-1)x^2-a^2$ and $x^3-(a^2-1)x-a$.

24. The L. C. M. of three or more quantities may be obtained by a similar process to that indicated in [22] for the G. C. M.

25. It appears from [20] that all the common measures of a and b are measures of d, i.e. d is a common *multiple* of all of them, and, being itself one of them, is therefore their L. C. M. We see also that all the common measures of a and b are comprised in such terms of the series d, $\frac{1}{2}d$, $\frac{1}{3}d$, &c. as are *integers*, and (70) all the common multiples of a and b in the terms of the series m, $2m$, $3m$, &c. Also in (64) we have $c=rd$, $b=qc+d=(qr+1)d$ $a=pb+c=(pqr+p+r)d$, which expressions exhibit the actual quotients, $pqr+p+r$, $qr+1$, when a and b are divided by d.

26. The results of (85) may be applied to produce many remarkable consequences in fractions, as follows.

Thus, if $\dfrac{a+x}{a}=\dfrac{m}{n}$, then $\dfrac{x}{a}=\dfrac{m-n}{n}$, $\dfrac{x+a}{x-a}=\dfrac{m}{m-2n}$;

if $\dfrac{a-x}{x}=m$ or $\dfrac{m}{1}$, then $\dfrac{a}{x}=\dfrac{m+1}{1}$, $\dfrac{a}{a+x}=\dfrac{m+1}{m+2}$;

if $\dfrac{a+x}{a-x}=\dfrac{(m+n)^2}{(m-n)^2}$, then $\dfrac{(a+x)+(a-x)}{(a+x)-(a-x)}=\dfrac{(m+n)^2+(m-n)^2}{(m+n)^2-(m-n)^2}$,

or $\dfrac{2a}{2x}=\dfrac{2(m^2+n^2)}{4mn}$, that is, $\dfrac{a}{x}=\dfrac{m^2+n^2}{2mn}$;

whence $\dfrac{a+x}{x}=\dfrac{(m+n)^2}{2mn}$, $\dfrac{a-x}{a}=\dfrac{(m-n)^2}{m^2+n^2}$, $\dfrac{a}{a+x}=\dfrac{m^2+n^2}{(m+n)^2}$, &c.

So (86), if $\dfrac{x}{a}=5=\dfrac{5}{1}$, then $\dfrac{2x}{2x-3a}=\dfrac{10}{10-3}=\dfrac{10}{7}$, $\dfrac{a+3x}{2x-a}=\dfrac{16}{9}$, &c.:

and again (87), if $\dfrac{x}{a}=\dfrac{1}{2}$, then $\dfrac{3x^2+2a^2}{a^2-2x^2}=\dfrac{3+8}{4-2}=\dfrac{11}{2}$;

if $\dfrac{x}{a}=-\dfrac{1}{2}=\dfrac{-1}{2}$ or $\dfrac{1}{-2}$, then $\dfrac{3x^3+2a^3}{a^3-2x^3}=\dfrac{-3+16}{8+2}=\dfrac{3-16}{-8-2}=\dfrac{13}{10}$.

27. But the results of (85-87) are only particular cases of a yet more general principle, which may be thus stated.

If $\dfrac{a}{b}=\dfrac{c}{d}$, then any fraction whatever, formed by means of a and b, with num$_r$ and den^r *homogeneous*, will be equal to a similar fraction, formed by means of c and d.

To prove this, let $\dfrac{a}{b}=\dfrac{c}{d}=x$; then $a=bx$, $c=dx$: now if in the first of two such fractions we put bx for a, and in the second dx for c, we shall find (since their terms are *homogeneous*) that b may be struck out altogether from the first and d from the second, and each fraction (since they are similar in form) will become the *same* function of x only: hence they are equal to each other.

Thus $\dfrac{a^3+2a^2b}{ab^2-3b^3}=\dfrac{b^3(x^3+2x^2)}{b^3(x-3)}=\dfrac{x^3+2x^2}{x-3}=\dfrac{c^3(x^3+2x^2)}{c^3(x-3)}=\dfrac{c^3+2c^2d}{cd^2-3d^3}$.

Also since $\dfrac{a}{c}=\dfrac{b}{d}$, the same is, of course, true if we form similar homogeneous fractions with a and c, and with b and d.

Ex. 3.

1. If $\frac{a}{b}=3$, write down the values of $\frac{a+b}{b}$, $\frac{a-b}{a}$, $\frac{a-2b}{b}$, $\frac{a+3b}{a}$.

2. If $\frac{a+b}{a-b}=\frac{1}{2}$, write down the values of $\frac{a}{b}$, $\frac{a-3b}{b}$, $\frac{2a}{3a-b}$, $\frac{2a+3b}{5b-a}$.

3. If $2x=y$, obtain the values of $\frac{x+2y}{y}$, $\frac{4x^2}{3y^2}$, $\frac{x^2+2y^2}{y^2}$, $\frac{y^3}{2x^3-y^3}$.

4. If $\frac{x}{y}=-\frac{1}{3}$, obtain the values of $\frac{3x^2+2y^2}{4x^2}$, $\frac{x^2-y^2}{x^2+y^2}$, $\frac{2x^3-3y^3}{2y^3-3x^3}$.

5. If $3a = 2b$, write down the values of

$$\frac{a^2-3ab}{b^2-3ab},\quad \frac{a^3+b^3}{a^2b-ab^2},\quad \frac{2a^3-a^2b+b^3}{a^2b+ab^2+2b^3},\quad \frac{a^4-3a^2b+2b^4}{(a^2-b^2)^2}.$$

6. If $2x+y=0$, write down the values of

$$\frac{x^3+2y^3}{x^2-xy},\quad \frac{x^2-2xy+3y^2}{3x^2+2xy-y^2},\quad \frac{x^3-x^2y+xy^2}{(x+y)^3},\quad \frac{x(x^2+y^2)+(x^2+y^2)^2}{x^3(2x+y)+(x-y)^2y}.$$

---

28. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=$ &c. $=x$ suppose, it may be easily shewn that any two fractions are equal, which are so formed that the numr and denr of the one are *any* homogeneous functions of a, c, e, &c., and those of the other, similar functions of b, d, f, &c., or else, the two numrs of a, c, e, &c., and the two denrs of b, d, f, &c.

$$\text{Thus } \frac{a^3+c^3}{ace+e^3}=\frac{x^3(b^3+d^3)}{x^3(bdf+f^3)}=\frac{b^3+d^3}{bdf+f^3},\quad \frac{ac}{bd}=x^2=\frac{a^2+ae-c^2}{b^2+bf-d^2}.$$

29. $\frac{a+c+e}{b+d+f}$ lies between the *greatest* and *least* of $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$.

For let $\frac{a}{b}$, the *greatest*, $=x$, and $\frac{e}{f}$, the *least*, $=y$;

$\therefore a=bx,\ c<dx,\ e<fx$, and $(a+c+e)<(b+d+f)x$;

again $a>by,\ c>dy,\ e=fy$, and $(a+c+e)>(b+d+f)y$;

$\therefore \frac{a+c+e}{b+d+f}<x$ and $>y$, that is, it lies between $\frac{a}{b}$ and $\frac{e}{f}$.

In like manner it may be shewn that the fraction whose numr and denr are *any* homogeneous functions of n dimensions, with *terms all positive*, of a, c, e, and of b, d, f, respectively, lies between the greatest and least of the fractions $\frac{a^n}{b^n}$, $\frac{c^n}{d^n}$, $\frac{e^n}{f^n}$; and the same may be proved of any number of fractions.

Ex. 4.

1. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{b}=\frac{a+c}{b+d}=\frac{a+mc}{b+md}$, $\frac{a}{c}=\frac{a-b}{c-d}=\frac{ma-b}{mc-d}$.

2. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{c^2}{d^2}=\frac{a^2+c^2}{b^2+d^2}=\frac{(a-c)^2}{(b-d)^2}$, $\frac{b^2}{d^2}=\frac{(a+mb)^2}{(c+md)^2}=\frac{a^2-b^2}{c^2-d^2}$.

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that

3. $\frac{a}{b}=\frac{c+e}{d+f}=\frac{ma-ne}{mb-nf}=\frac{a-c-e}{b-d-f}=\frac{ma-nc-pe}{mb-nd-pf}=\frac{m(a+e)-nc}{m(b+f)-nd}$.

4. $\frac{a^2}{b^2}=\frac{a^2-e^2}{b^2-f^2}=\frac{(a+c)^2}{(b+d)^2}=\frac{mc^2-ne^2}{md^2-nf^2}=\frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{(a-mc+ne)^2}{(b-md+nf)^2}$.

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30. If in the fraction $\frac{2x^3-2}{x^2-1}$ we give x the value 1, the num^r and den$_r$ will become each of them *zero*, and the fraction will assume the *illusory* form $0 \div 0$. Fractions which take this form when a particular value is given to some letter in the num^r and den^r, are called *Vanishing Fractions*. The value of such a fraction is altogether *indeterminate*.

This peculiarity, however, generally arises from the num$_r$ and den^r having a *common factor*, which factor becoming *zero*, when the particular value is given to the variable letter, makes each of them vanish: and if this be detected and struck out, the resulting fraction will have a definite value.

Thus $\frac{2(x^3-1)}{x^2-1}=\frac{2(x-1)(x^2+x+1)}{(x-1)(x+1)}=\frac{2(x^2+x+1)}{x+1}=3$, when $x=1$.

The result thus obtained is usually called the *value* of the Vanishing Fraction, which is said to be *evaluated*. More strictly, however, the value of the Fraction is, as above said, indeterminate; and 3 should be called the *Limit of the value of* $\frac{2(x^3-1)}{x^2-1}$, *as x becomes more nearly* $=1$: for, in the first place, it is absurd to speak of one actual zero being triple of another, and, besides, when the factor $x-1=0$, we are not at liberty to treat it like a *finite* quantity, and strike it out. (The fallacy of such a step would be obvious, if we were to reason thus; since $1\times 0=0$ and $2\times 0=0$, therefore $1\times 0=2\times 0$, and, therefore, striking out the zero's, $1=2$).

If, however, $x = 1$ *very nearly*, there is no objection to our striking out the common factor $x - 1$, however small it may be, if not actually zero; and since the resulting fraction will become more and more nearly $= 3$, as x becomes more and more nearly $= 1$, so also will the original fraction, since it is obtained from the other by merely multiplying its num^r and den^r by the *same* finite, though very small factor, $x - 1$. And so it appears that, though by reason of this multiplication, the num^r and den^r of the original fraction become both very small quantities, and smaller and smaller as x approaches 1, yet the num^r becomes more and more nearly *triple* of the den^r.

Ex. 6.

Evaluate the following vanishing fractions:

1. $\dfrac{x - x^5}{1 - x^3}$, when $x = 1$.
2. $\dfrac{x^2 + 2x - 35}{x^2 - 6x + 5}$, when $x = 5$.
3. $\dfrac{x^2 + x - 6}{x^2 + 2x - 3}$, when $x = -3$.
4. $\dfrac{x^3 + 2x^2 - 4x - 8}{x^3 + 3x^2 - 4}$, when $x = -2$.
5. $\dfrac{1 - 6x + 5x^2}{1 - 4x - 5x^2}$, when $x = \frac{1}{5}$.
6. $\dfrac{x^3 - a^3}{x^3 - 2ax^2 + 2a^2x - a^3}$, when $x = a$.
7. $\dfrac{2x^3 - 7x^2 + 12}{x^3 - 7x + 6}$, when $x=2$.
8. $\dfrac{x^3 - 19x + 30}{x^3 - 5x^2 + 18}$, when $x = 3$.
9. $\dfrac{2x^2 - 3ax - 2a^2}{2x^3 + 3ax^2 - a^3}$, when $x = -\frac{1}{2}a$.
10. $\dfrac{3x^3 + 4x^2 + 10x + 3}{3x^3 - 2x^2 + 8x + 3}$, when $x = -\frac{1}{3}$.
11. $\dfrac{2}{x^2 - 1} - \dfrac{1}{x - 1}$, when $x=1$.
12. $\dfrac{a(x^2 + c^2) - 2acx}{b(x^2 + c^2) - 2bcx}$, when $x = c$.

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31. Just as in (117), when the den^r of a fraction is a *trinomial* surd, as $\sqrt{a} + \sqrt{b} + \sqrt{c}$, if we multiply both num^r and den^r by $\sqrt{a} + \sqrt{b} - \sqrt{c}$, the latter becomes $(\sqrt{a} + \sqrt{b})^2 - c = a + b - c + 2\sqrt{ab}$; and if we again multiply both num^r and den^r by $a + b - c - 2\sqrt{ab}$, the latter will become $(a + b - c)^2 - 4ab$, and will thus be rationalized.

$$\text{Thus } \frac{1}{4 - \sqrt{2} - \sqrt{3}} = \frac{4 - \sqrt{2} + \sqrt{3}}{(4 - \sqrt{2})^2 - 3} = \frac{(4 - \sqrt{2} + \sqrt{3})(15 + 8\sqrt{2})}{(15 - 8\sqrt{2})(15 + 8\sqrt{2})}$$
$$= \tfrac{1}{97}(44 + 17\sqrt{2} + 15\sqrt{3} + 8\sqrt{6}).$$

32. Again, let the den^r be *any* binomial of the form $\sqrt[p]{a} + \sqrt[q]{b}$.

Let $x = \sqrt[p]{a}$, $y = \sqrt[q]{b}$; then, if m be the L. C. M. of p and q, both x^m and y^m will be rational: now $(x + y)(x^{m-1} - x^{m-2}y + \&c.) = x^m \pm y^m$,

where the sign will be + or − according as m is odd or even; hence we have, as the rationalizing factor for $x+y$ or $\sqrt[p]{a}+\sqrt[q]{b}$,

$$x^{m-1}-x^{m-2}y+x^{m-3}y^2-\&c. = a^{\frac{m-1}{p}}-a^{\frac{m-2}{p}}b^{\frac{1}{q}}+a^{\frac{m-3}{p}}b^{\frac{2}{q}}-\&c.$$

In like manner, since $(x-y)(x^{m-1}+x^{m-2}y+\&c.)=x^m-y^m$, we may obtain the rationalizing factor, as before, for $\sqrt[p]{a}-\sqrt[q]{b}$.

Ex. In $a^{\frac{1}{3}}-b^{\frac{1}{2}}$ the L. C. M. is 6: and the rationalizing factor is $(a^{\frac{1}{3}})^5+(a^{\frac{1}{3}})^4(b^{\frac{1}{2}})+(a^{\frac{1}{3}})^3(b^{\frac{1}{2}})^2+\&c. = a^{\frac{5}{3}}+a^{\frac{4}{3}}b^{\frac{1}{2}}+ab+a^{\frac{2}{3}}b^{\frac{3}{2}}+a^{\frac{1}{3}}b^2+b^{\frac{5}{2}}$, and the rationalized quantity is $(a^{\frac{1}{3}})^6-(b^{\frac{1}{2}})^6=a^2-b^3$.

Ex. 6.

Express with rational denominators.

1. $\dfrac{1+\sqrt{2}}{1+\sqrt{2}+\sqrt{3}}$. 2. $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$. 3. $\dfrac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$. 4. $\dfrac{3+2\sqrt{5}}{1-\sqrt{3}+\sqrt{5}}$.

Write down the rationalizing factors, and rationalized results, for

5. $\sqrt[3]{2}+\sqrt[3]{3}$. 6. $\sqrt{a}+\sqrt[3]{b}$. 7. $3^{\frac{1}{2}}-5^{\frac{2}{3}}$. 8. $a^{\frac{1}{2}}+x^{\frac{3}{4}}$. 9. $x-y^{\frac{2}{3}}$. 10. $x^{\frac{1}{3}}+y^{\frac{1}{4}}$.

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33. Unless a and b are such that a^2-b is a perfect square, and so $\sqrt{a^2-b}$ rational, the result of (119) will be useless; since the values there obtained for $\sqrt{x}$ and $\sqrt{y}$, being each a *complex* surd (that is, a surd within a surd,) the expression $\sqrt{x}\pm\sqrt{y}$ will be more complicated than the original *single* complex surd, $\sqrt{a\pm\sqrt{b}}$. But if $a^2-b=c^2$, a *perfect square*, we have $\sqrt{a\pm\sqrt{b}}=\sqrt{\frac{1}{2}(a+c)}\pm\sqrt{\frac{1}{2}(a-c)}$, or the given *complex* surd = the sum of two *simple* ones.

The following are a few additional and more difficult Examples.

Ex. 7.

Find the square roots of

1. $a^2+2x\sqrt{a^2-x^2}$. 2. $1+\sqrt{1-m^2}$. 3. $2a+2\sqrt{a^2-b^2}$

4. $mn-2m\sqrt{mn-m^2}$. 5. $a+b+c+2\sqrt{ac+bc}$. 6. $2+2(1-x)\sqrt{1+2x-x^2}$.

---

34. If the given surd be of the form $\sqrt{a^2c}+\sqrt{b}$ or $a\sqrt{c}+\sqrt{b}$, it may be written $\sqrt{c}\,(a+\sqrt{\frac{b}{c}})$, and then, if $a^2-\frac{b}{c}$ be a *perfect square*, the root of this quantity may be expressed in the form

$$\sqrt[4]{c}\,(\sqrt{x}+\sqrt{y})=\sqrt[4]{cx^2}+\sqrt[4]{cy^2},\ \text{that is, in the form } \sqrt[4]{\alpha}+\sqrt[4]{\beta}.$$

Ex. $\sqrt{27}+\sqrt{24}=3\sqrt{3}+2\sqrt{6}=\sqrt{3}\,(3+2\sqrt{2})$: here the *criterion* is satisfied for $3+2\sqrt{2}$, and the root required is $\sqrt[4]{3}\,(1+\sqrt{2})=\sqrt[4]{3}+\sqrt[4]{12}$.

35. We may sometimes succeed in extracting the square root of a quantity of the form $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$ by assuming it $=\sqrt{x}+\sqrt{y}+\sqrt{z}$:

then $a+\sqrt{b}+\sqrt{c}+\sqrt{d}=x+y+z+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{yz}$;

where we may put $2\sqrt{xy}=\sqrt{b}$, $2\sqrt{xz}=\sqrt{c}$, $2\sqrt{yz}=\sqrt{d}$, and if the values thus obtained satisfy also $x+y+z=a$, we have the root required.

Ex. Let $\sqrt{6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}}=\sqrt{x}+\sqrt{y}+\sqrt{z}$:

then $2\sqrt{xy}=2\sqrt{2}$, $2\sqrt{xz}=2\sqrt{3}$, $2\sqrt{yz}=2\sqrt{6}$,

whence $x=1$, $y=2$, $z=3$, which satisfy $x+y+z=6$;

$\therefore$ the root required is $1+\sqrt{2}+\sqrt{3}$.

Find the square roots of Ex. 8.

1. $\sqrt{18}-\sqrt{16}$. 2. $3\sqrt{6}+2\sqrt{12}$. 3. $8\sqrt{3}-6\sqrt{5}$. 4. $2+\frac{1}{2}\sqrt{18}$.
5. $\sqrt{27}+2\sqrt{6}$. 6. $4\sqrt{3}-\sqrt{21}$. 7. $3\sqrt{5}+2\sqrt{10}$. 8. $5\sqrt{2}-2\sqrt{12}$.
9. $9+4\sqrt{2}+4\sqrt{3}+2\sqrt{6}$. 10. $10-2\sqrt{3}-2\sqrt{6}+6\sqrt{2}$.
11. $10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}$. 12. $1\frac{5}{6}-\sqrt{2}-\frac{1}{3}\sqrt{6}+\frac{2}{3}\sqrt{3}$.

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36. *If* $\sqrt[3]{a+\sqrt{b}}=x+\sqrt{y}$, then $\sqrt[3]{a-\sqrt{b}}=x-\sqrt{y}$.

For $a+\sqrt{b}=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y}$; $\therefore a=x^3+3xy$, $\sqrt{b}=3x^2\sqrt{y}+y\sqrt{y}$,
and $a-\sqrt{b}=x^3-3x^2\sqrt{y}+3xy-y\sqrt{y}$; $\therefore \sqrt[3]{a-\sqrt{b}}=x-\sqrt{y}$.

37. *To extract the cube root of* $a\pm\sqrt{b}$.

Assume a quantity m, and suppose that $\sqrt[3]{m(a+\sqrt{b})}=x+\sqrt{y}$;
then $\sqrt[3]{m(a-\sqrt{b})}=x-\sqrt{y}$, and by Mult^n, $\sqrt[3]{m^2(a^2-b)}=x^2-y$;
now take m^2 such that $m^2(a^2-b)$ may be a *perfect cube* $=c^3$;

then $x^2-y=c$ (i); but, as in [37], $ma=x^3+3xy$ (ii);

$\therefore 4x^3-3cx=ma$, from which equation if a value of x can be found by trial, then $y=x^2-c$, and $\sqrt[3]{a\pm\sqrt{b}}=(x\pm\sqrt{y})\div\sqrt[3]{m}$.

If a^2-b be a perfect cube, then m^2 may be taken $=1$.

Ex. Find the cube root of $38+17\sqrt{5}$.

Let $\sqrt[3]{m(38+17\sqrt{5})}=x+\sqrt{y}$; then $\sqrt[3]{m(38-17\sqrt{5})}=x-\sqrt{y}$,
and $\sqrt[3]{m^2(1444-1445)}=\sqrt[3]{m^2(-1)}=x^2-y$:
put $m=1$; then $x^2-y=\sqrt[3]{-1}=-1$ (i), and $x^3+3xy=38m=38$ (ii);
$\therefore 4x^3+3x=38$; and, by trial, $x=2$ is a root of this equation,

$\therefore y=x^2+1=5$, and $\sqrt[3]{38\pm17\sqrt{5}}=2\pm\sqrt{5}$.

Or thus: obtain m as before, and put [37] $(3x^2+y)\sqrt{y}=m\sqrt{b}$, in which equation it will generally be very easy to guess at $\sqrt{y}$, which must either be the surd-factor in $\sqrt{b}$, or some multiple of it.

Thus, in the above Ex., $(3x^2+y)\sqrt{y}=17\sqrt{5}$, where, taking $\sqrt{y}=\sqrt{5}$, then $3x^2+y=17$, and $x=2$, as before.

38. *If* $\sqrt[n]{\sqrt{a}+\sqrt{b}} = \sqrt{x}+\sqrt{y}$, *then* $\sqrt[n]{\sqrt{a}-\sqrt{b}} = \sqrt{x}-\sqrt{y}$.

For $\sqrt{a}+\sqrt{b} = (\sqrt{x}+\sqrt{y})^n = x^{\frac{1}{2}n} + nx^{\frac{1}{2}(n-1)}y^{\frac{1}{2}} + \frac{1}{2}n(n-1)x^{\frac{1}{2}(n-2)}y + \&c.$ Here (i) if n be *even*, the odd terms of this expansion will be all *rational*, and the even *irrational;* therefore the given equality could not exist at all, unless one of the two quantities $\sqrt{a}$ or $\sqrt{b}$, (suppose $\sqrt{a}$,) be rational, $= a'$ suppose, and the other irrational; in which case a' = sum of odd terms, $\sqrt{b}$ = sum of even terms, and therefore $a' - \sqrt{b} = (\sqrt{x}-\sqrt{y})^n$, or $\sqrt[n]{\sqrt{a}-\sqrt{b}} = \sqrt{x}-\sqrt{y}$.

(ii) If n be *odd*, the odd terms involve $\sqrt{x}$ and the even $\sqrt{y}$; hence the given equality could only exist by one of the two former quantities, ($\sqrt{a}$ suppose,) having the same surd factor as $\sqrt{x}$, (including the case of $\sqrt{a}$ and $\sqrt{x}$ being both *rational*,) and $\sqrt{b}$ the same as $\sqrt{y}$; in which case also, as before, $\sqrt[n]{\sqrt{a}-\sqrt{b}} = \sqrt{x}-\sqrt{y}$.

39. *To extract, when possible, the* n^{th} *root of* $\sqrt{a} \pm \sqrt{b}$.

We shall suppose a and b both integers, since every case may be reduced to this form by (112); and n to be an *odd* number, since, in other cases, we may extract the square root of the given quantity, and the square root of this again, and so on, until the index is left an odd number.

Let $\sqrt[n]{m(\sqrt{a}+\sqrt{b})} = \sqrt{x}+\sqrt{y}$; then [38] $\sqrt[n]{m(\sqrt{a}-\sqrt{b})} = \sqrt{x}-\sqrt{y}$:

$\therefore\ x - y = \sqrt[n]{m^2(a-b)} = c$ suppose, if m be taken such that $m^2(a-b)$ may be a perfect n^{th} power $= c^n$:

also $\sqrt[n]{m^2(\sqrt{a}+\sqrt{b})^2} + \sqrt[n]{m^2(\sqrt{a}-\sqrt{b})^2} = 2(x+y)$, which, as might be shewn, will always be an *integral* quantity in all those cases in which the root can be obtained in the form of a binomial surd: hence if s be the integer next *greater* than one of these surds, and t the integer next *less* than the other, we have $s + t = 2(x+y)$, which equation, with $x - y = c$, gives x and y;

$$\text{and}\quad \sqrt[n]{\sqrt{a}\pm\sqrt{b}} = \left\{\sqrt{s+t+2c} \pm \sqrt{s+t-2c}\right\} \div 2\sqrt[n]{m}.$$

Ex. Let $\sqrt[5]{m(41+29\sqrt{2})} = \sqrt{x}+\sqrt{y}$; $\therefore \sqrt[5]{m^2(1681-1682)} = x - y$: put $m = 1$; $\therefore x - y = -1$; and, squaring and adding the first equations, $2(x+y) = \sqrt[5]{3363+2378\sqrt{2}} + \sqrt[5]{3363-2378\sqrt{2}} = \sqrt[5]{6725.991} + \sqrt[5]{.008}$ $= (5+f) + (1-f) = 6$; $\therefore x = 1$, $y = 2$, and $\sqrt[5]{41+29\sqrt{2}} = 1+\sqrt{2}$.

Ex. 9.

1. $\sqrt[3]{10\sqrt{7}+22}$. 2. $\sqrt[3]{16+8\sqrt{5}}$. 3. $\sqrt[3]{7-5\sqrt{2}}$. 4. $\sqrt[3]{10-6\sqrt{3}}$.
5. $\sqrt[3]{3-\frac{11}{9}\sqrt{6}}$. 6. $\sqrt[3]{2\frac{1}{2}-\frac{7}{4}\sqrt{2}}$. 7. $\sqrt[3]{\frac{13}{18}\sqrt{6}-\frac{5}{4}\sqrt{2}}$. 8. $\sqrt[3]{24\sqrt{21}-64}$.
9. $\sqrt[3]{25+22\sqrt{2}}$. 10. $\sqrt[3]{11+5\sqrt{7}}$. 11. $\sqrt[5]{232+164\sqrt{2}}$. 12. $\sqrt[5]{76+44\sqrt{3}}$.

40. *Imaginary* quantities are treated like ordinary surds: but one thing must be observed with respect to them of some importance. Since the square root of a quantity is that quantity which, *being squared, will produce the given one*, it follows that the square of $\sqrt{-a}$ will produce $-a$, that is, $\sqrt{-a} \times \sqrt{-a} = -a$. But by Mult^n of surds we have $\sqrt{-a} \times \sqrt{-a} = \sqrt{-a \times -a} = \sqrt{+a^2} = \pm a$; and how is this difference to be explained?

The fact is that, whenever we write $\sqrt{a^2} = \pm a$, we mean that, as a^2 *might* have arisen from squaring either $+a$ or $-a$, so here we have no reason to prefer one root to the other. If, however, as in the case before us, we know that it *has* arisen from squaring $-a$, of course we take only the negative root.

41. So also $\sqrt{-a} \times \sqrt{-b} = \sqrt{(-a) \times (-b)}$, or $\sqrt{ab}$, but this may be $+\sqrt{ab}$ or $-\sqrt{ab}$; writing, however, $\sqrt{-a}$ and $\sqrt{-b}$ in the forms $\sqrt{a} \times \sqrt{-1}$ and $\sqrt{b} \times \sqrt{-1}$, we have their product $\sqrt{ab} \times (\sqrt{-1})^2$ or $-\sqrt{ab}$, which determines, therefore, the sign of $\sqrt{-a} \times \sqrt{-b}$.

It is often best to express imaginary quantites in the form just employed for $\sqrt{-a}$ and $\sqrt{-b}$, viz. by means of the factor $\sqrt{-1}$; and it should be noticed that $\sqrt{-1} = \sqrt{-1}$, $(\sqrt{-1})^2 = -1$, $(\sqrt{-1})^3 = -\sqrt{-1}$, $(\sqrt{-1})^4 = +1$; and, generally, $(\sqrt{-1})^{4m} = +1$, $(\sqrt{-1})^{4m+1} = +\sqrt{-1}$, $(\sqrt{-1})^{4m+2} = -1$, $(\sqrt{-1})^{4m+3} = -\sqrt{-1}$, &c.

42. Of course our previous reasonings apply to *impossible*, as well as other, surd quantities; thus (118, iii) if $a + \sqrt{-b} = x + \sqrt{-y}$, then $a = x$, $b = y$, or if $a + b\sqrt{-1} = 0$, then $a = 0$, $b = 0$.

Ex. 1. $2\sqrt{-3} \times 3\sqrt{-2} = 2\sqrt{3}\sqrt{-1} \times 3\sqrt{2}\sqrt{-1} = -6\sqrt{6}$,

$$\frac{6\sqrt{-3}}{2\sqrt{-4}} = \frac{6\sqrt{3}\sqrt{-1}}{2\sqrt{4}\sqrt{-1}} = 3\sqrt{\frac{3}{4}} = \frac{3}{2}\sqrt{3}.$$

Ex. 2. $$\frac{4+\sqrt{-2}}{2-\sqrt{-2}} = \frac{(4+\sqrt{-2})(2+\sqrt{-2})}{(2-\sqrt{-2})(2+\sqrt{-2})} = \frac{8+6\sqrt{-2}-2}{4-(-2)}$$
$$= \tfrac{1}{6}(6+6\sqrt{-2}) = 1+\sqrt{-2}.$$

Ex. 3. Prove that $\sqrt{4+3\sqrt{-20}} + \sqrt{4-3\sqrt{-20}} = 6$.

Let $\sqrt{4+3\sqrt{-20}} = \sqrt{x} + \sqrt{y}$, $\therefore \sqrt{4-3\sqrt{-20}} = \sqrt{x} - \sqrt{y}$; and $\sqrt{16+180} = x - y$, $\therefore x - y = 14$; but (118, iii) $x + y = 4$; therefore, $x = 9$, $y = -5$, and $\sqrt{4 \pm 3\sqrt{-20}} = 3 \pm \sqrt{-5}$, from which the required result is evident.

Ex. 4. $(a+b\sqrt{-1})(a-b\sqrt{-1})=a^2+b^2$.

We see from this Ex. how, by means of impossible quantities, we may resolve a^2+b^2 into simple factors: also since a^2 and b^2 are each essentially positive (whether a and b be positive or negative) if $a^2+b^2=0$, we must have separately $a=0$, $b=0$, the same as we may obtain by putting either of its factors $a+b\sqrt{-1}$, or $a-b\sqrt{-1}$, $=0$.

Ex. 5. $(a+b\sqrt{-1})(c+d\sqrt{-1})=(ac-bd)+(ad+bc)\sqrt{-1}=a'+b'\sqrt{-1}$, suppose, which is a quantity of the same form as the original factors;

$\therefore (a+b\sqrt{-1})(c+d\sqrt{-1})(e+f\sqrt{-1})=(a'+b'\sqrt{-1})(e+f\sqrt{-1})=a''+b''\sqrt{-1}$, &c.;

that is, the product of any number of factors of the form $a+b\sqrt{-1}$ will be of the same form; except in the case of Ex. 4, where the impossible part has disappeared, its coefficient being $ab-ab$ or zero.

Ex. 6. $$\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}}=\frac{(a+b\sqrt{-1})(c-d\sqrt{-1})}{c^2+d^2}=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}\sqrt{-1}$$
$$=a'+b'\sqrt{-1}.$$

Reduce **Ex. 10.**

1. $2\sqrt{-4}+3\sqrt{-9}+5\sqrt{-16}$, and $2\sqrt{-12}+\dfrac{1}{\sqrt{-3}}-3\sqrt{-4\tfrac{11}{16}}$,
2. $(1+\sqrt{-1})^2+(1-\sqrt{-1})^3+(1+\sqrt{-1})^4+(1-\sqrt{-1})^5$.
3. $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$, $\dfrac{3-3\sqrt{-1}}{2+2\sqrt{-1}}$, $(x+1)\left(x-\dfrac{1+\sqrt{-3}}{2}\right)\left(x-\dfrac{1-\sqrt{-3}}{2}\right)$.
4. $\left(\dfrac{1+\sqrt{-2}}{1-\sqrt{-2}}+\dfrac{1-\sqrt{-2}}{1+\sqrt{-2}}\right)\times\left(\dfrac{1+\sqrt{-2}}{1-\sqrt{-2}}-\dfrac{1-\sqrt{-2}}{1+\sqrt{-2}}\right)$.
5. $(x-a\sqrt{-1})(x+a\sqrt{-1})\{x+\tfrac{1}{2}a(\sqrt{3}+\sqrt{-1})\}\times$
$\{x+\tfrac{1}{2}a(\sqrt{3}-\sqrt{-1})\}\{x-\tfrac{1}{2}a(\sqrt{3}+\sqrt{-1})\}\{x-\tfrac{1}{2}a(\sqrt{3}-\sqrt{-1})\}$
6. Find the fourth power of $-\sqrt{(-2\sqrt{-3})}$, and of $-\sqrt{(-\tfrac{3}{2}\sqrt{-4\tfrac{8}{9}})}$ and the square of $\sqrt{x+\sqrt{-y^2}}+\sqrt{x-\sqrt{-y^2}}$.
7. Write down the square, cube, and fourth powers of $a+b\sqrt{-1}$; and shew that they are of the same form.
8. Find the square roots of $31+12\sqrt{-5}$, and $-24\sqrt{-1}-7$; and the fourth root of -64.

CHAPTER II.

SIMPLE, QUADRATIC, AND OTHER EQUATIONS.

43. If we had the equation $3 + x + \sqrt{(x^2 + 9)} = 2$, (which differs from (120. Ex. 2) only in the sign before the root) we might proceed to solve it thus:

$$1 + x = -\sqrt{(x^2 + 9)};\ \therefore\ 1 + 2x + x^2 = x^2 + 9, \text{ and } x = 4 \text{ as before.}$$

But now, if we put the value 4 for x in the given equation, we shall find that though the former equation (with $-$ before the root) is satisfied, the latter (with $+$ before the root) is *not*. And, in fact, it may be easily shewn that there is no quantity that will satisfy the two equations; for, if there be, adding the two equations together, we have $6 + 2x = 4$, or $x = -1$, which will be found to satisfy *neither* of them. The student then will notice that, in all similar cases, the value of x obtained from the *squared* equation, will not satisfy *both* its *square roots:* it can only be ascertained by trial to which of them it really belongs. We know no means of finding the root of $3 + x + \sqrt{(x^2 + 9)} = 2$, if it have any.

44. A simple equation of one unknown quantity may always be reduced to the form $ax + b = 0$, or $a(x - \alpha) = 0$, if we put $\alpha = -\dfrac{b}{a}$.

Now (i) if a and b are both *finite*, then $x = -\dfrac{b}{a}$ is the value of the root required: (ii) if a be finite, and $b = 0$, then $x = 0$: (iii) if $a = 0$, and b be finite, then $x = \infty$, that is, there is in this case *no* finite value of x, which will satisfy the equation, and we see plainly that the smaller a is, the larger x must be, so that the product of a and x may equal any finite quantity b: (iv) if $a = 0$, and $b = 0$, then $x = \frac{0}{0}$, that is, its value is *indeterminate* [42], and, in fact, it is plain that $0 \times x = 0$, for *all* finite values of x.

45. Such cases as the above will often occur in the solution of problems, which are given in general terms, as will be seen in the discussion of the following.

Prob. 1. *What number must be added to* b *to produce* a?

Let x be the number: then $b + x = a$, or $x = a - b$.

Now, if $a > b$, $a - b$ will be *positive*, and the Problem, as it stands,

will be satisfied by this value of x; thus if $a = 48$, $b = 36$, we have $x = 48 - 36 = 12$, and 12 is the number which *added* to 36 will produce 48.

But, if $a < b$, $a - b$ will be *negative;* how are we to explain this result? We must remember that the Algebraical meaning of the word *Addition* is more extensive than the common, or Arithmetical, and that the term is used in Algebra for the bringing into one sum any number of quantities, whether positive or negative, so that we speak of *adding* a positive quantity, or of *adding* a negative quantity, as $a + (+ 2a) = 3a$, $a + (- 2a) = - a$. Now when we expressed the above Problem in algebraical language, we could not restrict the word *add* to its ordinary meaning: the algebraical will indeed include the ordinary, but will include more, and the equation $b + x = a$ will really give the solution of this more general question, 'What number must be *added algebraically*, or joined to b to produce a?'—taking in the case of the original number being *greater* as well as *less* than the other, in which case the quantity to be *added* (algebraically) will be a negative quantity.

In such a case, the ordinary statement of the Problem will be absurd; thus, it would be absurd to ask, 'What number must be *added* to 48 to produce 36?' We must either understand the word 'added' in its more extended sense, or modify the statement of the Problem, thus: 'What number must be *taken* from 48, &c.?'

If we choose to do the former, we should have $48 + x = 36$, and $x = - 12$, which *added* (algebraically) to 48 produces 36; but, if we modify the question, we should have $48 - x = 36$, and $x = 12$, which *taken* from 48 produces 36.

And, generally, if in any case we obtain a *negative* root, the proper Arithmetical form, into which the question should be modified so as to produce the same root *positive*, will be suggested by writing $-x$ for x in the equation, and observing the result.

PROB. 2. *A's age is* a *years, B's is* b *years: when will A be three times as old as B?*

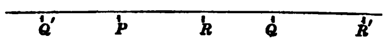
Let x = number of years required; then $a + x = 3(b + x)$, and $x = \frac{1}{2}(a - 3b)$. Now if $a > 3b$, the value of x is *positive*, and the answer is obtained to the Problem, as it stands: thus if A's age be 36, and B's 8, we have $x = \frac{1}{2}(36 - 24) = 6$, and we see that, in 6 years, A's age will be 42 and B's 14.

But if $a < 3b$, the value of x will be negative; writing $-x$ for x

in the equation we have $a - x = 3(b - x)$, which suggests that the question should now be put in the form, 'When *was* A &c.,' the algebraical expression $+ x$ having, as before, included the idea of x being *negative* as well as *positive*, that is, of the required number of years having to be *taken from* as well as *added to* the given ages: thus if $a = 36$, and $b = 14$, we have $x = \frac{1}{2}(36 - 42) = -3$, and 3 years *back* A was aged 33 and B 11.

Sometimes an algebraical solution to a Problem may have no arithmetical interpretation at all, consistent with the practical possibilities of the Problem. Thus if, in the above, $b > a$, it is of course impossible that A's age should ever be, or have been, triple of B's: yet still the algebraical solution is obtained as before, $x = \frac{1}{2}(a - 3b)$; thus, if $a = 10$, $b = 12$, then $x = \frac{1}{2}(10 - 36) = -13$. The fact is that, all along, the algebraical root has really had nothing whatever to do with A and B or their ages, but is merely a solution of the general question, *What number must be added* (algebraically) *to* a *and* b, *to make the former sum triple of the latter?* This will, indeed, include such a case as that supposed in the Problem about A and B, and all possible modifications of it besides; but is yet more comprehensive still.

PROB. 3. A and B travel in the same direction at the rate of a and b miles an hour respectively; and n hours after A reaches a certain point P, B reaches a certain point Q, c miles beyond it: when will they be together?



Let x = number of hours *after* A passes P that they will be together at R: then A will have gone ax miles beyond P, and B, having yet $n - x$ hours to travel before he reaches Q, will be $b(n - x)$ miles from Q; but the whole distance PQ is c miles:

$$\therefore\ ax + b(n - x) = c, \quad \text{whence } x = \frac{c - nb}{a - b}.$$

(i) Let A travel fastest, or $a > b$, so that $a - b$ is positive: then if $c > nb$, the value of x will be positive, that is, they will be together, as we supposed, sometime *after* A has passed P. [This is plain also from the Problem; for when A is at P, B (who in n hours is to be at Q) must be somewhere *in* PQ, since the whole distance PQ, or c, $> nb$, that is, $>$ the distance B would travel in n hours.] In this case A, travelling faster, will overtake B. If c be

also $> na$, then $x > \dfrac{na - nb}{a - b} > n$; or A will not overtake B till after the first n hours, that is, till after B has passed Q: if $c = na$, he will overtake him in n hours exactly, that is, at the point Q; and if $c < na$ but still $> nb$, x will be $< n$, and he will overtake him before he reaches Q. If $c = nb$, then $x = 0$, or A will pass B at P. [In this case, since the distance PQ is nb, B must have been at P at the same moment as A, in order to be at Q n hours after.] If $c < nb$, then x becomes *negative*, or the required number of hours is to be *taken away* from that when A is at P, instead of being *added to* it; in other words A *had* passed B, before reaching P, and the question, to be arithmetical, should be modified by asking 'When *were* they together?'

(ii) Let a be $< b$, or B travel fastest, then $x = \dfrac{c - nb}{a - b} = \dfrac{nb - c}{b - a}$; which result may be discussed in the same way as the former.

(iii) Let $a = b$, then $x = \infty$, unless $c = nb$, when $x = \frac{0}{0}$: that is, if their rates be equal, they *never* can be together, unless the distance PQ be such that B can travel it in n hours exactly, in which case he will be together with A at P, and then they will *always* continue together.

46. Supposing we had the same Problem proposed, only that A and B are moving in *opposite* directions; then, proceeding as before, since in $n - x$ hours after they are together B is to be at Q, they must meet somewhere as at R' beyond Q from P: and $PR' = ax$, $QR' = b(n - x)$;

$$\therefore\ ax - b(n - x) = c, \text{ and } x = \frac{c + nb}{a + b}.$$

In this case x will always be positive, or they will always meet some time or other after A passes P, as will be obvious on a moment's consideration of the Problem.

We may observe, however, that the former algebraical solution will include this case also, by merely changing b into $-b$, which expresses that B's motion is *negative*, or in the *opposite* direction, with respect to A's.

In like manner also, if, in any of the above results, we change c into $-c$, we shall get the corresponding answers for cases in which the distance c is not now to be measured in the direction PQ of

A's motion, as we supposed originally, but in the *opposite* direction, PQ': and so too, the change of n into $-n$ will supply the answers to cases in which B is supposed to reach Q n hours *before*, instead of *after*, A reaches P.

47. We will here explain the method of solving certain questions by the Arithmetical Rules of Single and Double Position.

(i) *Single Position.*—This may be applied, whenever the question could have been stated algebraically in the form $ax = b$, as, for instance, when the sum of any given parts of some unknown number is to be equal to a certain given number.

In such a case, assuming any known number n, perform the statement of the question upon it, and let the result be c;

$$\text{then } ax = b,\ an = c, \text{ and } \therefore \frac{x}{n} = \frac{b}{c}, \text{ or } x = \frac{b}{c}n.$$

Ex. *Find a number such that, when increased by its fourth and diminished by its fifth part, the result shall be* 63.

Assuming for n any number, as 20, we have $20 + 5 - 4 = 21$:

$$\therefore x = \tfrac{63}{21} \times 20 = 60.$$

(ii) *Double Position.*—This may be applied, whenever the question could have been stated algebraically in the form $ax + b = c$.

In such a case, perform the statement of the question upon any two known numbers, n, n', and let the errors in the result be e, e', both (suppose) in excess:

$$\text{then } \left.\begin{aligned} an + b &= c + e \\ an' + b &= c + e' \\ ax + b &= c \end{aligned}\right\} \quad \begin{aligned} &\therefore a(n - x) = e, \quad a(n' - x) = e'; \\ &\text{and } \frac{n - x}{n' - x} = \frac{e}{e'}, \quad \text{or } x = \frac{n'e - ne'}{e - e'}. \end{aligned}$$

Ex. *Find a number such that, when it is divided by* 9, *and the quotient diminished by* 3, *three times the remainder will be* 30.

Here $3(\frac{1}{9}x + 3) = 30$: assume $n = 9$, $n' = 18$; then, performing the operations indicated, we have

$$\text{(i) } 3(1 - 3) = -6, \text{ and } e = -36, \quad \text{(ii) } 3(2 - 3) = -3, \text{ and } e' = -33;$$

$$\therefore x = \frac{18(-36) - 9(-33)}{-36 + 33} = 117.$$

It is obvious, however, that such questions would be much more easily solved by the common processes of *Simple Equations*.

48. The methods in (95-99) are the best for practical purposes; but the following, called the Method of *Indeterminate Multipliers*, is well worthy of notice for its elegance.

$$\left.\begin{aligned} ax + by &= c \\ a'x + b'y &= c' \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Multiply (2) by an indeterminate quantity l, and add it to (1); then $(a + la')\,x + (b + lb')\,y = c + lc'$: $\qquad (\alpha)$

now put $b + lb' = 0$, so that $lb' = -b$, and multiply (α) by b';

$$\text{then} \quad (ab' - ba')\,x = cb' - bc', \quad \text{or } x = \frac{cb' - bc'}{ab' - ba'}:$$

similarly, putting $a + la' = 0$, and multiplying (α) by a',

$$\text{we have } (ba' - ab')\,y = ca' - ac', \quad \text{or } y = \frac{ca' - ac'}{ba' - ab'}.$$

49. In the above results, the following cases may be noticed:

(i) If $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$, then $ab' = a'b$, $ac' = a'c$, $bc' = b'c$: in this case, x and y assume the form $\frac{0}{0}$, and their values are *indeterminate*. It will be found, in fact, that in this case, there are not two *independent* equations, one of them being only a multiple of the other: for let $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = m$; then $a = ma'$, $b = mb'$, $c = mc'$; and therefore (1) becomes $m\,(a'x + b'y) = mc'$, or $a'x + b'y = c'$, identical with (2). There being then only one equation between x and y, if we give *any* value to x or y, there will be a corresponding value for y or x.

(ii) If $\frac{a}{a'} = \frac{b}{b'}$ only, then $ab' = a'b$; and $x = \frac{cb' - bc'}{0} = \infty$, $y = \infty$: that is, there are *no* finite values of x and y which will satisfy the equations, which in this case will be found to be *incompatible* with each other: for, as before, let $a = ma'$, $b = mb'$; then (1) becomes $ma'x + mb'y = c$, or $a'x + b'y = \frac{c}{m}$, which is incompatible with (2), since the same expression $a'x + b'y$ is assumed equal to two *different* quantities, which is absurd.

(iii) If $\frac{a}{a'} = \frac{c}{c'}$ only, then $ac' - a'c = 0$, and, consequently, $y = 0$;

$$\text{while } x = \frac{cb' - bc'}{ab' - ba'} = \frac{c}{a}\cdot\frac{b' - b\,\frac{c'}{c}}{b' - b\,\frac{a'}{a}} = \frac{c}{a}, \quad \left(\text{since } \frac{c'}{c} = \frac{a'}{a},\right) \quad \text{or } = \frac{c'}{a'},$$

as appears also from (1) or (2) by giving to y its value *zero*.

50. The following is also solved by Indeterminate Multipliers:

$$\left.\begin{array}{l} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{array}\right\} \quad \begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Here $x = \dfrac{d_1 + ld_2 + md_3}{a_1 + la_2 + ma_3}$, if $\begin{cases} b_1 + lb_2 + mb_3 = 0 & (\alpha) \\ c_1 + lc_2 + mc_3 = 0 & (\beta) \end{cases}$

whence $b_1c_3 - b_3c_1 + (b_2c_3 - b_3c_2)\,l = 0$, or $l = \dfrac{b_3c_1 - b_1c_3}{b_2c_3 - b_3c_2}$,

and $b_1c_2 - b_2c_1 + (b_3c_2 - b_2c_3)\,m = 0$, or $m = \dfrac{b_1c_2 - b_2c_1}{b_2c_3 - b_3c_2}$;

$$\therefore\ x = \frac{d_1(b_2c_3 - b_3c_2) + d_2(b_3c_1 - b_1c_3) + d_3(b_1c_2 - b_2c_1)}{a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)}.$$

The law of formation of the above value of x is very simple. The coefficients of d_1, d_2, d_3, in the num^r are the same as those of a_1, a_2, a_3, in the den^r: that of d_1, viz. $b_2c_3 - b_3c_2$, is formed of the coeff^s of y and z in equations (2) and (3), multiplied across, viz. b_2 into c_3 and b_3 into c_2; that of d_2 is similarly formed out of equations (3) and (1), and that of d_3 out of equations (1) and (2). [*Observe*, not (1) and (3), but (3) and (1), keeping always the order of the figures 1, 2, 3, 1, 2, 3, &c.]

The value of y or z may now be written down by *Symmetry*, the coeff^s of d_1, d_2, d_3, being the same as those of b_1, b_2, b_3, in the value of y, or of c_1, c_2, c_3, in that of z, respectively; and formed out of the coeff^s of x and z, or of x and y, by the same Law as before.

$$\text{Thus } y = \frac{d_1(a_2c_3 - a_3c_2) + d_2(a_3c_1 - a_1c_3) + d_3(a_1c_2 - a_2c_1)}{b_1(a_2c_3 - a_3c_2) + b_2(a_3c_1 - a_1c_3) + b_3(a_1c_2 - a_2c_1)},$$

$$z = \frac{d_1(a_2b_3 - a_3b_2) + d_2(a_3b_1 - a_1b_3) + d_3(a_1b_2 - a_2b_1)}{c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)}.$$

51. If in the above the equations for finding l and m are *incompatible*, then [49. ii] $b_2c_3 - b_3c_2 = 0$, and in this case x may be found from the two equations involving these quantities:

thus $(2) \times b_3 - (3) \times b_2$ gives $x(a_2b_3 - a_3b_2) = d_2b_3 - d_3b_2$.

If they are *insufficient*, then [49. i] $b_1c_3 - b_3c_1 = 0$, $b_1c_2 - b_2c_1 = 0$; and x may be similarly found from any two of the given equations.

52. If the three equations in [50] are equivalent to only *two* equations, then one of them, as (3), must be the sum of some multiples of the other two; so that we may write

$$a_3 = ma_1 + na_2,\quad b_3 = mb_1 + nb_2,\quad c_3 = mc_1 + nc_2,\quad d_3 = md_1 + nd_2:$$

whence, by eliminating m and n between the *first three* of these,

$$a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)=0;$$

so also, by eliminating m and n between the *last three*,

$$d_1(b_2c_3-b_3c_2)+\&c.=0;\ \therefore x=\frac{0}{0};\ \text{and similarly } y=\frac{0}{0},\ z=\frac{0}{0}.$$

53. Any equation, in which the unknown quantity is found in two terms, and with its index in one twice as great as in the other, that is, any equation which can be put into the form $ax^{2m}+bx^m+c=0$, may be solved as a quadratic equation.

Ex. 1. $x^4-13x^2=-36$. Here $x^4-13x^2+(\frac{13}{2})^2=\frac{169}{4}-36=\frac{25}{4}$ whence $x^2=\frac{13}{2}\pm\frac{5}{2}$, $=9$ or 4; $\therefore x=\pm 3$, or ± 2: and thus the given equation, which is a *biquadratic*, admits of the *four* roots, $+2$, -2, $+3$, -3.

Ex. 2. $x^{-1}+x^{-\frac{1}{2}}=2$. Here $x^{-1}+x^{-\frac{1}{2}}+\frac{1}{4}=\frac{9}{4}$, and $x^{-\frac{1}{2}}+\frac{1}{2}=\pm\frac{3}{2}$;

$$\therefore x^{-\frac{1}{2}}=1 \text{ or } -2, \text{ and } x^{\frac{1}{2}}=1 \text{ or } -\tfrac{1}{2};\ \therefore x=1 \text{ or } \tfrac{1}{4}.$$

54. Still more generally, any equation which can be put into the form

$$A(ax^{2m}+bx^m+c)^2+B(ax^{2m}+bx^m+c)+C=0$$

may be treated as a quadratic, of which the roots are the values of $ax^{2m}+bx^m+c$, whence x may be found.

Ex. 1. $x^2+6\sqrt{(x^2-2x+5)}=11+2x$.

Observing the quantity under the root, we may put this in the form

$$(x^2-2x+5)+6\sqrt{(x^2-2x+5)}+9=11+5+9=25;$$

$\therefore \sqrt{(x^2-2x+5)}=-3\pm5=2$ or -8; whence (i) $x^2-2x+5=4$, and $x=1, 1$, or (ii) $x^2-2x+5=64$, and $x=1\pm2\sqrt{15}$.

Here again, as in [43], the latter roots do not belong to the equation as given, but to that obtained by taking the radical with a different sign. In general, we give *all* the roots, without stopping to enquire which may suit the particular form that may be given us.

Ex. 2. $4x^4-4x^3-27x^2+14x=392$;

Here, completing the square with the first two terms,

$$(4x^4-4x^3+x^2)-28x^2+14x=392;$$

or $(2x^2-x)^2-14(2x^2-x)-392=0$, whence $2x^2-x=28$ or -14;

$\therefore$ (i) $2x^2-x-28=0$, and $x=4$ or $-3\frac{1}{2}$,

(ii) $2x^2-x+14=0$, and $x=\frac{1}{4}\{1\pm\sqrt{-111}\}$.

Ex. 3. $(a+x)^{\frac{2}{3}} + 4(a-x)^{\frac{2}{3}} = 5(a^2-x^2)^{\frac{1}{3}}$.

Here $\left(\dfrac{a+x}{a-x}\right)^{\frac{2}{3}} - 5\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{3}} = -4$, whence $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{3}} = 4$ or 1;

$$\therefore \frac{a+x}{a-x} = 64 \text{ or } 1, \text{ and } x = \tfrac{63}{65}a \text{ or } 0.$$

Ex. 11.

1. $x^4 - 5x^2 + 4 = 0$. 2. $x^{-2} - 2x^{-1} = 8$. 3. $(x^2-2)^2 = \frac{1}{4}(x^2+12)$.

4. $2\sqrt{x} + 2x^{-\frac{1}{2}} = 5$. 5. $x^{-\frac{3}{4}} + 27x^{\frac{3}{4}} = 28$. 6. $(x^2-9)^2 = 3 + 11(x^2-2)$.

7. $(x^2-1)(x^2-2) + (x^2-3)(x^2-4) = x^4 + 5$. 8. $4x^{\frac{2}{3}}(x^{\frac{2}{3}} - 2) = 9x^{\frac{2}{3}} - 4$.

9. $(x^{-\frac{1}{3}} + 2)(x^{-\frac{2}{3}} + 5) = x^{-1} + 8$. 10. $(x+1)^3 - x^2(x^2-1) = (x-1)^3 + 2(x^2+3)$.

11. $\dfrac{x^{-n}}{1+x^{-n}} + \dfrac{1-x^{-n}}{x^{-n}} = \dfrac{1}{6}$. 12. $(x^2+x+1)^2 - \frac{1}{4}(x^2+3) = 2x(x+1)^2$.

13. $\sqrt[4]{(x+7)} + \frac{3}{4}\sqrt{(x+7)} = 5$. 14. $5x - 7x^2 - 8\sqrt{(7x^2-5x+1)} = 8$.

15. $x^2 - 3x + 7\sqrt{(11x-2x^2+2)} = \frac{5}{2}x + 21$. 16. $2x^2 - 2x + 2\sqrt{(2x^2-7x+6)} = 5x - 6$.

17. $3x(3-x) = 11 - 4\sqrt{(x^2-3x+5)}$. 18. $x + \sqrt{(x^2-ax+b^2)} = a^{-1}x^2 + b$.

19. $x^2 - x + 5\sqrt{(2x^2-5x+6)} = \frac{3}{2}(x+11)$. 20. $9x - 4x^2 + \sqrt{(4x^2-9x+11)} = 5$.

21. $x + 4 + \sqrt{\dfrac{x+4}{x-4}} = \dfrac{12}{x-4}$. 22. $(1+x)^{\frac{2}{5}} + \frac{3}{16}(1-x)^{\frac{2}{5}} = (1-x^2)^{\frac{1}{5}}$.

23. $\sqrt[2mn]{x^{m+n}} = \dfrac{a^2-b^2}{2(a^2+b^2)}(\sqrt[m]{x} + \sqrt[n]{x})$. 24. $a^2b^2x^{\frac{1}{n}} - 4(ab)^{\frac{3}{2}}x^{\frac{m+n}{2mn}} = (a-b)^2x^{\frac{1}{m}}$.

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55. Here let us notice some particular cases of the roots of the equation $ax^2 + bx + c = 0$.

(1) If $c = 0$, then $ax^2 + bx = 0$, and (133) $x = 0$ or $-\dfrac{b}{a}$:

(2) If $b = 0$, then $ax^2 = -c$, and $x = \pm\sqrt{-\dfrac{c}{a}}$:

(3) If $a = 0$, write $\dfrac{1}{y}$ for x; then $a + by + cy^2 = 0$, or $by + cy^2 = 0$;

$$\therefore y = 0 \text{ or } -\frac{b}{c}, \text{ and } x = \frac{1}{y} = \infty \text{ or } -\frac{c}{b}:$$

(4) If $b = c = 0$, then $ax^2 = 0$, and $x = \pm 0$:

(5) If $a = b = 0$, then $cy^2 = 0$, $\therefore y = \pm 0$ and $x = \pm\infty$:

(6) If $a = c = 0$, then $bx = 0$ or $x = 0$, and $by = 0$ or $y = 0$, that is, $x = \infty$, so that in this case the roots are $x = 0$ and ∞:

(7) If $a = b = c = 0$, then $x = \frac{0}{0}$, and the roots are indeterminate.

Such cases as these will occur in the solution of *general* problems, when the constants involved in the algebraical statements of them have certain particular values given them. In this way it may happen that the coeff. of x^2 will disappear, and the equation, though really a quadratic, assume the form of a simple one: as, *e.g.* in $(p-q)x^2 - qx - p = 0$, whose roots are $p \div (p-q)$ and -1, if we take the particular case of $p=q$, the coeff. of x^2 disappears, and the roots become $p \div 0$ and -1, or ∞ and -1. In such a case the root ∞ may be considered as the *Limit* to which the value of x approaches, as the coeff. of x^2 approaches to *zero*.

56. (130) Or thus: Since α, β are roots of $x^2+px+q=0$,

$$\therefore \alpha^2+p\alpha+q=0, \qquad \beta^2+p\beta+q=0:$$

subtracting, $\alpha^2-\beta^2+p(\alpha-\beta)=0$, or $\alpha+\beta+p=0$, whence $-p=\alpha+\beta$;

$$\text{and } q=-p\alpha-\alpha^2=\alpha^2+\alpha\beta-\alpha^2=\alpha\beta.$$

57. Similarly, let α, β, γ be the roots of $x^3+px^2+qx+r=0$:

$\therefore \alpha^3+p\alpha^2+q\alpha+r=0, \quad \beta^3+p\beta^2+q\beta+r=0, \quad \gamma^3+p\gamma^2+q\gamma+r=0;$

whence $\alpha^3-\beta^3+p(\alpha^2-\beta^2)+q(\alpha-\beta)=0$, or $\alpha^2+\alpha\beta+\beta^2+p(\alpha+\beta)+q=0$,

and $\alpha^3-\gamma^3+p(\alpha^2-\gamma^2)+q(\alpha-\gamma)=0$, or $\alpha^2+\alpha\gamma+\gamma^2+p(\alpha+\gamma)+q=0$;

$\therefore$ subtracting again,

$\alpha(\beta-\gamma)+\beta^2-\gamma^2+p(\beta-\gamma)=0$, or $\alpha+\beta+\gamma+p=0$, whence $-p=\alpha+\beta+\gamma$;

$\therefore q=-\alpha^2-\alpha\beta-\beta^2-p(\alpha+\beta)=\alpha\beta+\alpha\gamma+\beta\gamma, \ r=-\alpha^3-p\alpha^2-q\alpha=\alpha\beta\gamma.$

And so on, with equations of higher order; but the above method of proof can only be applied when the roots are *unequal*, since, otherwise, the divisors, $\alpha-\beta$, &c. become *zero*.

58. If α, β, γ, &c. be the roots of $x^n+p_1x^{n-1}+$ &c. $=0$, (or, as we shall write it, of $f(x)=0$), then

$$f(x)=(x-\alpha)(x-\beta)(x-\gamma)\ldots\ldots$$

Suppose $f(x)$ divided by $x-\alpha$, until the remainder R no longer contains x; then, if Q represent the quotient, $f(x)=Q(x-\alpha)+R$, for all values of x: now put $x=\alpha$, then $f(\alpha)=0$, since α is a root of $f(x)=0$; and $\therefore 0=0+R$, whence $R=0$, or $f(x)$ is divisible without remainder by $x-\alpha$.

Similarly, $f(x)$ is divisible by each of the n factors, $x-\alpha$, $x-\beta$, $x-\gamma$, &c., and, consequently, by their product $(x-\alpha)(x-\beta)(x-\gamma)\ldots$; but this product has its first term x^n, the same as $f(x)$; hence $f(x)$ cannot contain any other factors but these, that is

$$f(x)=(x-\alpha)(x-\beta)(x-\gamma)\ldots$$

By comparing this result with (180), we see that

$p_1 = -\alpha - \beta - \gamma - \&c,$ or $-p_1 = \alpha + \beta + \gamma + \&c,$

$p_2 = (-\alpha)(-\beta) + (-\alpha)(-\gamma) + \&c,$ or $p_2 = \alpha\beta + \alpha\gamma + \beta\gamma + \&c,$

$p_3 = (-\alpha)(-\beta)(-\gamma) + \&c,$ or $-p_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \&c,$

and, $(-1)^r p_r =$ sum of products of the roots taken r together.

The above proof applies, however, only to the case of unequal roots, but the statement itself is generally true, as shewn in Treatises on the Theory of Equations: hence if $f(x)$ have m roots each $=\alpha$, n roots each $=\beta$, &c, we shall have $f(x) = (x-\alpha)^m (x-\beta)^n \ldots$ and $-p_1 = m\alpha + n\beta + \&c,$ with other corresponding modifications.

59. If an equation can be thrown into the form $(x-a)X=0$, (X representing some function of x,) it may be satisfied by putting either $x - a = 0$, or $X = 0$: the first of these gives $x = a$, but the second, being solved, will also give roots of the original equation.

And, generally, if an equation can be separated into factors, functions of x, the roots, obtained by equating each of these factors to zero, will be roots of the original equation.

60. Hence we may sometimes solve a cubic equation, or one of a higher order, whenever we can either detect by inspection one or more of the roots, or can separate the equation into factors.

Ex. 1. Find the three cube roots of unity.

Let $\alpha = \sqrt[3]{1}$, then $\alpha^3 = 1$, or $\alpha^3 - 1 = 0$; now this equation, being evidently satisfied by $\alpha = 1$, will be divisible by $\alpha - 1$; $\therefore \alpha^2 - \alpha + 1 = 0$, whence $\alpha = \frac{1}{2}(1 \pm \sqrt{-3})$, which are the other two roots.

Hence also the roots of $x^3 - a^3 = 0$ are $x = a$, $x = \frac{1}{2}(1 \pm \sqrt{-3})\,a$.

Ex. 2. Solve the equation $x^4 - x^3 - x^2 + x = 0$.

This may be thrown into the form

$$x^3(x-1) - x(x-1) = (x^3 - x)(x-1) = x(x^2-1)(x-1) = 0,$$

the factors of which being x, $x+1$, $x-1$, $x-1$, we have the roots $0, -1, +1, +1$.

61. There is no general method of solving equations higher than quadratic, except by processes of considerable complexity, and these only of limited application. There is one species, however, of such Equations, which deserves more particular notice.

Def. An equation is said to be *reciprocal*, which is not altered by changing x into $\frac{1}{x}$.

This will be found to be the case, whenever the coeff^s of its terms, equidistant from beginning and end, are either *equal* and of the *same* sign, or else *equal* and of *different* signs, provided also, that *the middle term be wanting*, if the equation be of an *even* degree, and the corresponding coeff^s of *different* signs. On this account such equations are also called *recurring* equations, as in Ex. 1, 2.

Now it will be found, as below, that every reciprocal equation of *odd* degree will be divisible by $x-1$ or $x+1$, according as the last term is -1 or $+1$ (and will have a root, therefore, $+1$ or -1); and every reciprocal equation of *even* degree, *with its last term* -1, will be div. by x^2-1 (and will have roots, therefore, $+1$ and -1): and, in every case, the reduced equation after the div^n will be found to be still reciprocal, of an *even* degree, and with its last term $+1$.

By this means a reciprocal *cubic* may be reduced to a *quadratic*, and one of the *fifth* or *sixth* degree to a *biquadratic*, which latter may always be solved as follows.

Ex. 1. $x^5+2x^4-3x^3-3x^2+2x+1=0$.

Here, if we write x^{-1} for x, we get $x^{-5}+2x^{-4}-3x^{-3}-3x^{-2}+2x^{-1}+1=0$, or, multiplying each side by x^5, $\quad 1+2x-3x^2-3x^3+2x^4+x^5=0$, the same equation as before.

This may be written $(x^5+1)+2x(x^3+1)-3x^2(x+1)=0$,

where each bracket is plainly divisible by $x+1$;

dividing, therefore, the original equation by $x+1$, we get

$$x^4+x^3-4x^2+x+1=0, \quad \text{or } (x^4+1)+x(x^2+1)=4x^2;$$

$\therefore$, completing squares,

$$(x^4+2x^2+1)+x(x^2+1)+\tfrac{1}{4}x^2=4x^2+2x^2+\tfrac{1}{4}x^2=\tfrac{25}{4}x^2;$$

$\therefore\ x^2+1+\frac{1}{2}x=\pm\frac{5}{2}x$, and $\therefore$ (i) $x^2-2x+1=0$, (ii) $x^2+3x+1=0$: from (i) $x=1, 1$, from (ii) $x=\frac{1}{2}(-3\pm\sqrt{5})$; so that, including also the root -1, corresponding to the factor $x+1$, we have the *five* roots of the given equation, -1, 1, 1, $\frac{1}{2}(-3\pm\sqrt{5})$.

Ex. 2. $x^6-x^5+x^4-x^2+x-1$.

This may be written $(x^6-1)-x(x^4-1)+x^2(x^2-1)=0$, where, each bracket being div. by x^2-1, we get (by such division)

$$x^4-x^3+2x^2-x+1=0, \quad \text{or } (x^4+1)-x(x^2+1)=-2x^2;$$

$\therefore$ completing squares, as before,

$$(x^2+1)^2-x(x^2+1)+\tfrac{1}{4}x^2=-2x^2+2x^2+\tfrac{1}{4}x^2=\tfrac{1}{4}x^2,$$

whence $x^2+1-\frac{1}{2}x=\pm\frac{1}{2}x$, and (i) $x^2-x+1=0$, (ii) $x^2+1=0$; from (i) $x=\frac{1}{2}(1\pm\sqrt{-3})$, from (ii) $x=\pm\sqrt{-1}$, which with $+1$, -1, for the factor x^2-1, are the *six* roots of the given equation.

62. Since a recurring equation is not altered by writing x^{-1} for x, it is plain that if $x=a$ be a root of it, so also will $x^{-1}=a$, or $x=a^{-1}$.

Hence we see that the roots of any reciprocal equation of an *even* degree with its last term $+1$, occur in pairs, the roots in each pair being the *reciprocals* of one another: thus, in Ex. 1, we have 1, the reciprocal of 1, and $\frac{1}{2}(-3+\sqrt{5})$ of $\frac{1}{2}(-3-\sqrt{5})$, as the Student may easily see. If, however, the equation be of an *odd* degree, it will have [61], besides these pairs of roots, a root $+1$ or -1, according as its last term is $+1$ or -1; or, if of an *even* degree, with its last term -1, then it will have the two roots ± 1.

These exceptions, however, do not, of course, *contradict* the statement at the beginning of this Article.

63. As the Solution of Equations affords excellent practice for the more advanced Student, here follow a miscellaneous collection. Some of these may be treated as reciprocal equations; others may be solved by guessing at a root, and reducing by division; in others artifices must be used, which will suggest themselves in practice. But it is impossible to lay down rules for all cases; the Student must be left here to exercise his own ingenuity.

Ex. 1. $\dfrac{a+2x+\sqrt{(a^2-4x^2)}}{a+2x-\sqrt{(a^2-4x^2)}}=\dfrac{5x}{a}$.

Here (85) $\dfrac{a+2x}{\sqrt{(a^2-4x^2)}}=\dfrac{5x+a}{5x-a}=\sqrt{\dfrac{a+2x}{a-2x}}$; $\therefore\ \dfrac{a+2x}{a-2x}=\left(\dfrac{5x+a}{5x-a}\right)^2$

and then again by (85) $\dfrac{2x}{a}=\dfrac{10ax}{25x^2+a^2}$, whence $x=\pm\frac{3}{5}a$.

Ex. 2. $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=\sqrt[3]{c}$.

Here, cubing by the formula $(a+b)^3=a^3+b^3+3ab(a+b)$, we have $(a+x)+(a-x)+3\sqrt[3]{(a^2-x^2)}\{\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}\}=2a+3\sqrt[3]{\{(a^2-x^2)c\}}=c$;

$$\therefore\ \sqrt[3]{\{(a^2-x^2)\}c}=\frac{c-2a}{3},\ \text{and}\ x=\pm\sqrt{\{a^2-\frac{(c-2a)^3}{27c}\}}.$$

Ex. 3. $1+x^4=a(1+x)^4=a(1+4x+6x^2+4x^3+x^4)$;

$\therefore\ (a-1)x^4+4ax^3+6ax^2+4ax+(a-1)=0$, a recurring equation;

$\therefore\ (a-1)\{x^2+x^{-2}\}+4a\{x+x^{-1}\}+6a=0=(a-1)\{x+x^{-1}\}^2+4a\{x+x^{-1}\}+4a+2$;

$$\therefore\ x+x^{-1}=\frac{-2a\pm\sqrt{\{2(a+1)\}}}{a-1}=p\ \text{suppose};$$

$$\text{then}\ x^2-px+1=0,\quad\text{or}\ x=\tfrac{1}{2}\{p\pm\sqrt{(p^2-4)}\};$$

where, by giving p its double value, we shall have the *four* roots.

Ex. 12.

1. $x^2 + x^{-2} + x + x^{-1} = 4.$ 2. $15x - 3x^2 + 4\sqrt{(x^2 - 5x + 5)} = 16.$

3. $x^2 + x^{-2} + x - x^{-1} = 2.$ 4. $x^5 - 1 = 0.$ 5. $x^3 - px = p - 1.$

6. $2x^3 - x^2 = 1.$ 7. $\sqrt[3]{(1+2x)} + \sqrt[3]{(1-2x)} = \sqrt[3]{c}.$ 8. $x^3 + px^2 + px + 1 = 0.$

9. $(n^2 + 1)(x^2 + a^2 - c^2) = 4nax.$ 10. $x^4 - x^3 = \frac{10}{36}x^2 + x - 1.$

11. $x^4 - 3x^3 = 1 - 3x.$ 12. $x^3 - 3x = 2.$ 13. $x - 1 = 2 + 2x^{-\frac{1}{2}}.$

14. $\sqrt[3]{(1+x)^2} - \sqrt[3]{(1-x)^2} = \sqrt[3]{(1-x^2)}.$ 15. $ax + 2\sqrt{(n^2x + nax^2)} = n(3x-1).$

16. $(a^{4m} + 1)(x^{\frac{1}{2}} - 1)^2 = 2(x + 1).$ 17. $x^3 - 2px^2 + 2px = 1.$

18. $x^4 + 3x + 1 = 3x^3 + \frac{4}{9}x^2.$ 19. $x^3 - \frac{4}{9}x^{-1} = 1\frac{4}{9}.$ 20. $x^{12} = 1.$

21. $x^4 + 1 = \frac{1}{2}(x^3 + 7x^2 - x).$ 22. $(1+x^5) = a(1+x)^5.$ 23. $2x^2 + \sqrt{(x^2+9)} = x^4 - 9.$

24. $4x^2 + 12x\sqrt{(1+x)} = 27(1+x).$ 25. $x^2 - 8(x+1)\sqrt{x} + 18x + 1 = 0.$

26. $\dfrac{\sqrt{x} + \sqrt{(x-a)}}{\sqrt{x} - \sqrt{(x-a)}} = \dfrac{n^2a}{x-a}.$ 27. $\dfrac{1 + x^4}{(1-x)^4} = a.$ 28. $\dfrac{a + x + \sqrt{(a^2 - x^2)}}{a + x - \sqrt{(a^2 - x^2)}} = \dfrac{c}{x}.$

29. $\dfrac{\sqrt{a} - \sqrt{\{a - \sqrt{(a^2 - ax)}\}}}{\sqrt{a} + \sqrt{\{a - \sqrt{(a^2 - ax)}\}}} = c.$ 30. $\dfrac{ax + 1 + \sqrt{(a^2x^2 - 1)}}{ax + 1 - \sqrt{(a^2x^2 - 1)}} = \dfrac{b^2x}{2}.$

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31. $\left.\begin{aligned} x + y &= 5 \\ x^4 + y^4 &= 97 \end{aligned}\right\}$ 32. $\left.\begin{aligned} x - y &= 2 \\ x^4 + y^4 &= 272 \end{aligned}\right\}$ 33. $\left.\begin{aligned} x^5 + y^5 &= a^5 \\ x + y &= b \end{aligned}\right\}$

34. $\left.\begin{aligned} \frac{1 + x}{1 - y} + \frac{1 + y}{1 - x} &= \frac{9}{13} \\ \frac{1 + x}{1 + y} + \frac{1 - y}{1 - x} &= \frac{4}{13} \end{aligned}\right\}$ 35. $\left.\begin{aligned} xy &= a(x + y) \\ x^2y^2 &= b^2(x^2 + y^2) \end{aligned}\right\}$

36. $\left.\begin{aligned} \sqrt{\frac{3x}{x + y}} + \sqrt{\frac{x + y}{3x}} &= 2 \\ xy - (x + y) &= 54 \end{aligned}\right\}$

37. $\left.\begin{aligned} 3x^2 + 4y^2 &= 7xy \\ x^{\frac{3}{2}} - \tfrac{2}{9}y^2 &= yx^{\frac{1}{2}} \end{aligned}\right\}$ 38. $\left.\begin{aligned} x^2 + x\sqrt[3]{(xy^2)} &= 208 \\ y^2 + y\sqrt[3]{(x^2y)} &= 1053 \end{aligned}\right\}$

39. $\left.\begin{aligned} \frac{x^2}{y^2} + \frac{2x + y}{\sqrt{y}} &= 20 - \frac{y^2 + x}{y} \\ x + 8 &= 4y \end{aligned}\right\}$ 40. $\left.\begin{aligned} \frac{x}{y} - \frac{y}{x} &= \frac{x + y}{x^2 + y^2} \\ \frac{x^2}{y^2} - \frac{y^2}{x^2} &= \frac{x - y}{y^2} \end{aligned}\right\}$

41. $\left.\begin{aligned} x^2 + xy + y^2 &= a^2 \\ x + \sqrt{(xy)} + y &= b \end{aligned}\right\}$ 42. $\left.\begin{aligned} x^my^n &= a^mb^nc \\ x^ny^m &= a^mb^nd \end{aligned}\right\}$

43. $\left.\begin{aligned} \sqrt{x} - \sqrt{y} &= \sqrt{x}(\sqrt{x} + \sqrt{y}) \\ (x + y)^2 &= 2(x - y)^2 \end{aligned}\right\}$ 44. $\left.\begin{aligned} \sqrt{(x-y)} + \tfrac{1}{2}\sqrt{(x+y)} &= \frac{x - 1}{\sqrt{(x-y)}} \\ x^2 + y^2 &= \tfrac{34}{15}xy \end{aligned}\right\}$

45. $x^ma^n + y^nb^m = 2\sqrt{\{(ax)^m(by)^n\}},\ xy = ab.$

46. $\left.\begin{aligned} x + y + z &= 14 \\ x^2 + y^2 + z^2 &= 84 \\ xz &= y^2 \end{aligned}\right\}$ 47. $\left.\begin{aligned} x^2y^2 &= c^3z \\ x^2z^2 &= b^3y \\ y^2z^2 &= a^3x \end{aligned}\right\}$ 48. $\left.\begin{aligned} x + y + z &= \tfrac{19}{6} \\ x^2 + y^2 + z^2 &= \tfrac{133}{36} \\ y^2 &= xz \end{aligned}\right\}$

49. $x^2 + y^2 + z^2 = 3037,\ y^2 + x = 871,\ y^2 + z = 877.$

50. $x + 2y + z = 19,\ x^2 + 4y^2 + z^2 = 133,\ x : 2y :: y : \frac{1}{2}z.$

64. (137) It is sometimes possible to *modify* the statement of a question, so as to admit the negative answers which would otherwise have to be rejected. In order to do this, we must, as before, change x into $-x$ in the equation which expresses the Problem, and consider the result. Thus, if we change x into $-x$, we get in (137 Ex. 1) $30 - x = x^2 - 12$, which indicates, for the negative root, this modified form of the question, 'What number, when *taken* from 30, &c.:' and, in like manner, in Ex. 2, we shall get $\frac{120}{-x+3} = \frac{120}{-x} - 2$, or $\frac{120}{x-3} = \frac{120}{x} + 2$, which indicates the modification of the question 'If he had bought 3 *less* for the same money, he would have paid £2 *more* for each.'

N.B. Additional Equations and Problems are given at the end.

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65. Any trinomial quantity, x^2+px+q, may be expressed in one of the three forms, $(x-\alpha)(x-\beta)$, $(x-\alpha)^2$, or $(x-\alpha)^2+\beta^2$, where α, β are *real* quantities. [The student will, of course, distinguish between the *quantity* x^2+px+q and the *equation* $x^2+px+q=0$.]

For $x^2+px+q = (x^2+px+\frac{1}{4}p^2)+(q-\frac{1}{4}p^2) = (x+\frac{1}{2}p)^2+(q-\frac{1}{4}p^2)$:

$\therefore$ (i) If $q < \frac{1}{4}p^2$, or $p^2 > 4q$, then $x^2+px+q = (x+\frac{1}{2}p)^2-(\frac{1}{4}p^2-q)$

$=\{x+\frac{1}{2}p+\sqrt{(\frac{1}{4}p^2-q)}\}\{x+\frac{1}{2}p-\sqrt{(\frac{1}{4}p^2-q)}\}=(x-\alpha)(x-\beta)$,

if $-\alpha=\frac{1}{2}p+\sqrt{(\frac{1}{4}p^2-q)}$, $-\beta=\frac{1}{2}p-\sqrt{(\frac{1}{4}p^2-q)}$;

(ii) If $q=\frac{1}{4}p^2$, or $p^2=4q$, then $x^2+px+q=(x+\frac{1}{2}p)^2=(x-\alpha)^2$, if $-\alpha=\frac{1}{2}p$;

(iii) If $q>\frac{1}{4}p^2$, or $p^2<4q$, then $x^2+px+q=(x-\alpha)^2+\beta^2$, if $\beta^2=q-\frac{1}{4}p^2$.

In (ii) the expression is a *perfect square*, and in (ii) and (iii) is necessarily *positive*, whatever value we give x.

Instead of saying that x^2+px+q may be put into one or other of the above forms according as $p^2 >, =, < 4q$, we may say (129) according as the *equation* $x^2+px+q=0$ has its roots *unequal*, *equal*, or *impossible*. If p^2-4q be a *perfect square*, then α, β will be also *rational*, or x^2+px+q can be resolved into *rational simple factors*.

In like manner, $ax^2+bx+c = a(x^2+\frac{b}{a}x+\frac{c}{a})$ may be put into one of the three forms $a(x-\alpha)(x-\beta)$, $a(x-\alpha)^2$, or $a\{(x-\alpha)^2+\beta^2\}$, according as $b^2 > = < 4ac$, or, in other words, according as the equation $ax^2+bx+c=0$ has its roots *unequal*, *equal*, or *impossible*. If it take the second form, the expression is (as before) a perfect square $=\{\sqrt{a}(x-\alpha)\}^2$; if the third, the factor $(x-\alpha)^2+\beta^2$ will be

positive for all values of x, and therefore the expression $ax^2 + bx + c$ will have always the same sign as a, that is, it will not *change its sign* whatever value we give to x. If $b^2 - 4ac$ be a *perfect square*, then $ax^2 + bx + c$ can be resolved into *rational simple factors*.

66. We shall conclude this chapter with a few remarks upon elimination in general.

If the number of equations given exceed by one, two, &c. the number of unknowns, these may be all eliminated, and there will arise one, two, &c. *equations of condition* between the coeff^{s} of the equations which must be satisfied, independently of the values of the unknowns, in order that the equations may be able to exist at all.

Ex. 1. $\left.\begin{array}{l} ax + b = 0 \\ a'x + b' = 0 \\ a''x + b'' = 0 \end{array}\right\}\begin{array}{l} (1) \\ (2) \\ (3) \end{array}$ Here, eliminating x from (1) and (2), from (1) and (3), and from (2) and (3), $ab' - a'b = 0$, $ab'' - a''b = 0$, $a'b'' - a''b' = 0$,
which equations, however, are not independent of each other, since from any two of them we may derive the third.

Ex. 2.
$$\left.\begin{array}{l} ax^2 + bx + c = 0 \\ a'x^2 + b'x + c' = 0 \end{array}\right\}\begin{array}{l} (1) \\ (2) \end{array}$$

Here eliminating x^2, we obtain $(a'b - ab')\,x + a'c - ac' = 0$;
eliminating c and c', and dividing by x, $(a'c - ac')\,x + b'c - bc' = 0$;
eliminating x from these, $(a'c - ac')^2 + (ab' - a'b)\,(b'c - bc') = 0$.

Ex. 3.
$$\left.\begin{array}{r} ax + by = c \\ a'x + b'y = c' \\ a''x + b''y = c'' \end{array}\right\}\begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

From (1) and (2) we get $x = \dfrac{b'c - bc'}{ab' - a'b}$, $y = \dfrac{ac' - a'c}{ab' - a'b}$;

$\therefore$ from (3) $a''\,(b'c - bc') + b''\,(ac' - a'c) + c''\,(a'b - ab') = 0$.

67. If there be two equations given of the second degree between two unknowns, the elimination of either of them will lead *generally* to an equation of the fourth degree.

Let $ax^2 + bxy + cy^2 + dx + ey + f = 0$, or $ax^2 + (by + d)\,x + (cy^2 + ey + f) = 0$,
$a'x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0$, $a'x^2 + (b'y + d')\,x + (c'y^2 + e'y + f') = 0$,
be two equations of the second degree in their most *general* form; then, eliminating x^2 and x, as in Ex. 2 above, we have

$$\{a'\,(cy^2 + ey + f) - a\,(c'y^2 + e'y + f')\}^2 + \&c. = 0,$$

which manifestly contains y^4, and is therefore of the fourth degree in y.

68. *To eliminate generally between two equations of two unknowns.*

Let $E_1=0$, $E_2=0$, represent any two equations in x and y; proceed with E_1, E_2, (arranged according to x) as if to find their G.C.M.; and let Y represent the remainder, when at length it becomes clear of x, and is therefore a function of y only. Now any value of y, which makes this rem^r vanish, that is, any root (β suppose) of the equation $Y=0$, will render the last divisor (64) a common factor of E_1 and E_2. Now, suppose we put β for y, throughout the operation; then all the quantities concerned would become functions of x only. Let us denote the new values of the last divisor, and of the two given quantities E_1, E_2, by X, X_1, X_2 respectively; then X_1 and X_2 will have the common factor X, and consequently, for any value α of x which makes $X=0$, we shall have $X_1=0$, $X_2=0$. It follows then that E_1, E_2, would be changed to X_1, X_2, by the value β of y; and each of X_1, X_2, to *zero* by the value α of x; so that $x=\alpha$, $y=\beta$, will satisfy the equations $E_1=0$, $E_2=0$.

Ex. 1. $\left.\begin{aligned} ax+by&=c \\ a'x+b'y&=c' \end{aligned}\right\}$ Here $\left.\begin{aligned} E_1&=ax+by-c=0 \\ E_2&=a'x+b'y-c'=0 \end{aligned}\right\}$

$$\begin{array}{l} ax+by-c)\ a'x+\ b'y-\ c' \\ \qquad\qquad\qquad\quad \underline{a\qquad\qquad\qquad} \\ \qquad\qquad\qquad\quad aa'x+ab'y-ac'\ (a' \\ \qquad\qquad\qquad\quad \underline{aa'x+a'by-a'c} \\ \qquad\qquad\qquad\quad (ab'-a'b)y-(ac'-a'c)=Y=0, \text{ if } y=\dfrac{ac'-a'c}{ab'-a'b}=\beta; \end{array}$$

and, by putting this value β for y, the last divisor $ax+by-c$ becomes $ax+\dfrac{a(bc'-b'c)}{ab'-a'b}$ or X, and $X=0$ gives $x=\dfrac{b'c-bc'}{ab'-a'b}$.

Ex. 2. $\left.\begin{aligned} x+y&=5 \\ x^2+y^2&=13 \end{aligned}\right\}$

$$\begin{array}{l} x+(y-5)\rfloor\ x^2+(y^2-13)\ \lfloor x-(y-5) \\ \qquad\qquad\quad\ \underline{x^2+(y-5)\,x\qquad\qquad} \\ \qquad\qquad\qquad -(y-5)\,x+(y^2-13) \\ \qquad\qquad\qquad \underline{-(y-5)\,x-(y^2-10y+25)} \\ \qquad\qquad\qquad\qquad\qquad 2y^2-10y+12=Y. \end{array}$$

Here $Y=0$ gives $y=2$, or $y=3$:
if $y=2$, we have $X=x+2-5=x-3$, and $X=0$ gives $x=3$;
if $y=3$, we have $X=x+3-5=x-2$, and $X=0$ gives $x=2$.

N.B. In the case of equations higher than the second degree, it would be necessary to see whether Y may not contain some factor, a function of y, introduced to avoid Fractions in the process for the G. C. M. Such a factor must be struck out of $Y=0$ before we use its roots, as above.

CHAPTER IV.

INDETERMINATE EQUATIONS, AND INEQUALITIES.

69. It is easily seen, by giving t the values 0, 1, 2, &c. successively, that the integral values of x and y in (138) form series in AR. PROG., whose common differences are b, a, respectively.

70. To shew that the number of *positive integral* solutions will be limited or not according as the equation is of the form (i) $ax + by = c$, or (ii) $ax - by = c$; and that in (i) the number of solutions will be the greatest integer either in $\frac{c}{ab}$ or in $\frac{c}{ab} + 1$.

For (i), when $ax + by = c$, we shall have $x = \alpha - bt$, $y = \beta + at$, where $a\alpha + b\beta = c$; and here we must not take t *positively* $> \alpha \div b$, or *negatively* $> \beta \div a$, if x and y are to be *positive* integers: hence if $\frac{\alpha}{b} = n + f$, $\frac{\beta}{a} = n' + f'$, where n, n' are the greatest integers in $\frac{\alpha}{b}$, $\frac{\beta}{a}$, and f, f' are proper fractions, the number of different values that may be given to t will be $n + n' + 1$, (adding 1 for the value $t = 0$, when $x = \alpha$, $y = \beta$); and, therefore, the number of solutions in positive integers will be $n + n' + 1$

$$= \frac{\alpha}{b} + \frac{\beta}{a} + 1 - (f + f') = \frac{a\alpha + b\beta}{ab} + 1 - (f + f') = \frac{c}{ab} + 1 - (f + f')$$

= the greatest integer in $\frac{c}{ab}$ or in $\frac{c}{ab} + 1$, according as the sum of the two fractions f and f' is $>$ or < 1.

Hence if $c < ab$, the equation cannot have more than one such solution, and may have none, which last must always be the case when $c < a + b$, (unless we allow the supposition of x or $y = 0$,) since then, if we give x and y their *least* integral values, viz. $x = 1$, $y = 1$, we have still $ax + by > c$.

And (ii) when $ax - by = c$, we have $x = \alpha + bt$, $y = \beta + at$, and here there is no limit in one direction to the value of t, and therefore the number of positive integral solutions is unlimited.

71. (140 Ex. 1. N.B.) If a be prime to b, there is always some multiple, pa, of a, where $p < b$, which is divisible by b with remainder *unity*.

For each term of the series a, $2a$, $3a$, &c. $(b-1)\,a$ when divided by b must leave a *different* remainder: if not, let ma and $m'a$ leave the same remainder r, so that we get $ma = nb + r$, $m'a = n'b + r$; then $(m - m')\,a = (n - n')\,b$, or $\frac{b}{a} = \frac{m - m'}{n - n'}$: now $\frac{b}{a}$ is in its lowest terms, since a is prime to b, and, consequently, $m - m'$, $n - n'$, must be equimultiples of b and a, which is here impossible, since m, m' are each less than b, and therefore $m - m'$ cannot be a multiple of b.

Hence these remainders, since they are each $< b$, and the n° of them is $b - 1$, must consist of all numbers from 1 to $b - 1$ inclusive; and, therefore, some one term pa, when divided by b, will give some quotient q with a remr 1, so that $pa = qb + 1$ for some value of $p < b$.

72. If $ax + by + cz = d$, we may write $ax + by = d - cz$, and then, giving z successive values, 1, 2, &c. we get a series of indeterminate equations of the form $ax + by = c'$.

Ex. Solve $2x + 7y + 5z = 27$ in *positive* integers.

We may transpose either of the three quantities to the second side; but it will be better to take that which will most limit the number of resulting equations: thus, if we write $2x + 7y = 27 - 5z$, we might take z from 1 to 5, and so we should form five equations; but if we write $2x + 5z = 27 - 7y$, we can only take y from 1 to 3; and here again [70 i] it is useless to take $y = 3$, since in that case we should have $2x + 5y = 6$, where $c < (a + b)$.

Hence (1) $2x + 5z = 20$, or (2) $2x + 5z = 13$:
from (1) $x = 10 - 5t$, $y = 1$, $z = 2t$; from (2) $x = 4 - 5t$, $y = 2$, $z = 2t + 1$.

In (1) we can only take $t = 1$, which gives $x = 5$, $y = 1$, $z = 2$; and in (2) we can only take $t = 0$, which gives $x = 4$, $y = 2$, $z = 1$.

Ex. 13.

1. Find the number of positive integral solutions of

$$3x + 5y + 7z = 100,\quad 5x + 7y + 11z = 144,\quad 17x + 19y + 21z = 200.$$

2. How many lbs. of tea at 3*s* 6*d*, 4*s* 6*d*, and 6*s* 6*d*, must be taken to make 30 lbs. at 6*s*?

3. The expenses of a party were 40*s*, of which each man paid 4*s*, each woman 3*s*, and each child 4*d*: how many were there at least?

4. How many hundred gals. of spirits, at 12*s*, 15*s*, and 18*s* per gallon, will make a mixture of 900 gals. at 17*s* per gallon?

5. In how many different ways can a refiner mix three kinds of silver of 7, 12, and 17 carats fine, so as to produce a mass of 50 oz. of 15 carats fine?

6. How much brandy at 15*s*, 16*s*, and 20*s* per gallon may be mixed with 10 gallons of water that the compound may be worth 11*s* 4*d* per gallon? Obtain the simplest answer the question admits of.

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73. Indeterminate equations of degree higher than the first are generally difficult of solution, and beyond the object of the present work. We will here, however, shew how to solve in integers an equation of the form $mxy + nx^2 = ax + by + c$, or $mxy + ny^2 = ax + by + c$, which involves only *one* of the squares of x and y.

Ex. $3xy + 2x^2 = 3x + 2y + 5$.

Here $y = \dfrac{-2x^2 + 3x + 5}{3x - 2} = \dfrac{-\frac{2}{9}z^2 + z + 5}{z - 2}$, if $z = 3x$;

$$\therefore\ 9y = \frac{-2z^2 + 9z + 45}{z - 2} = -2z + 5 + \frac{55}{z - 2} = -6x + 5 + \frac{55}{3x - 2};$$

let us take now all the divisors of 55, viz. ± 1, ± 5, ± 11, ± 55, and put $3x - 2$ equal to each of these which will give x an integer; thus we have

$$3x - 2 = +1,\ x = 1;\ 3x - 2 = -5,\ x = -1;\ 3x - 2 = -11,\ x = -3;$$
$$3x - 2 = +55,\ x = 19:$$

if $x = 1$, then $9y = 54$, $y = 6$; if $x = -1$, $y = 0$; if $x = -3$, $y = 2$; if $x = 19$, y is not integral:

hence the only solution in *positive* integers is $x = 1$, $y = 6$.

Of course the same method may be applied to an equation of the form $mxy = ax + by + c$.

Ex. 14.

Solve in positive integers,

1. $3xy = 5x + 2y + 4$.
2. $5xy + 2x - 3y = 42$.
3. $3xy + x + y = 79$.
4. $xy + 2x^2 = 2x + 3y + 29$.
5. $(x + 2)^2 = (x - 1)\,y$.
6. $x^2 + x + 5 = (3y - x)^2$.
7. $3x\,(y + 1) = 2\,(2y + 7)$.
8. $y\,(2x + 1) = 3x^2 + 1$.

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74. *Inequalities*, in which two expressions are connected by the sign $>$ or $<$, are treated just like equations, with this exception, that if we change the signs of *all* the terms, (as we may do in equations,) we must also change the sign of inequality from $>$ to $<$, or $<$ to $>$; thus, although 7 is > 3, yet -7 is < -3.

Most cases of inequality, however, will be found reducible to the following fact, viz. that, since $(x-y)^2$, or $x^2 - 2xy + y^2$, is essentially positive, whatever be the values of x and y, $\therefore\ x^2 + y^2 > 2xy$.

Ex. 1. Which is greater, $m^3 + 1$ or $m^2 + m$, whatever m may be?

$$\text{Here } m^3+1 \text{ is } > m^2 + m, \text{ if } m^3 + 1 > m(m+1),$$

or if $m^2 - m + 1 > m$, (dividing each side by $m+1$), or if $m^2+1 > 2m$, which being the case by [74], $\therefore\ m^3 + 1 > m^2 + m$.

Ex. 2. Shew that $a^3 + b^3 > a^2b + ab^2$.

$$\text{Since } a^2 + b^2 > 2ab, \ \therefore\ a^3 + ab^2 > 2a^2b \text{ and } a^2b + b^3 > 2ab^2;$$

$$\therefore\ a^3 + b^3 + a^2b + ab^2 > 2a^2b + 2ab^2, \text{ or } a^3 + b^3 > a^2b + ab^2.$$

Ex. 15.

1. Shew that 5 is the integer value of x, when
$$\tfrac{1}{4}(x+2) + \tfrac{1}{3}x < \tfrac{1}{2}(x-4) + 3 \text{ and } > \tfrac{1}{2}(x+1) + \tfrac{1}{3}.$$
2. Shew that the sum of any fraction, and its reciprocal, is > 2.
3. Shew that $\dfrac{a+b}{2} > \dfrac{2ab}{a+b}$, and that $\dfrac{a}{b^2} + \dfrac{b}{a^2} > \dfrac{1}{b} + \dfrac{1}{a}$.
4. Shew that $a^6 + a^4b^2 + a^2b^4 + b^6 > (a^3 + b^3)^2$.
5. If $x^2 = a^2 + b^2$, and $y^2 = c^2 + d^2$, shew that $xy > ac + bd$ or $ad + bc$.
6. If $a > b$, shew that $a^4 - b^4 < 4a^3(a-b)$ and $> 4b^3(a-b)$.
7. Shew that $a^2 + b^2 + c^2 > ab + ac + bc$.
8. Shew that $a^3 + b^3 + c^3 > \tfrac{1}{2}(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2)$.
9. Shew that $a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 > 6abc$.
10. Shew that $(a+b+c)^3 > 27abc$.
11. Shew that $2(1 + a^2 + a^4) > 3(a + a^3)$.
12. Shew that $abc > (a+b-c)(a+c-b)(b+c-a)$.

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75. An inequality of the second degree of one variable may always be reduced to the form (i) $x^2 + px + q > 0$, or (ii) $x^2 + px + q < 0$.

Now [65] the expression $x^2 + px + q$ may be made to assume one of three forms:

(1) when $x^2 + px + q = (x-\alpha)(x-\beta)$, (i) is satisfied by any value of x, which is $>$ the greatest of α and β, or $<$ the least; and (ii) by any value of x, intermediate to α and β;

(2) when $x^2 + px + q = (x - a)^2$, (i) is satisfied by any value of x, except $x = a$, and (ii) is impossible;

(3) when $x^2 + px + q = (x - a)^2 + \beta^2$, (i) is satisfied by any value of x, and (ii) is impossible.

76. Equations may sometimes be applied to find *maxima* or *minima* values of some given function of a quantity, by which we mean the *greatest* or *least* values which such a function can possibly have, while different values are given to the quantity itself.

Thus the *minimum* value of $3x^2 - 3x + 2$ will be found to be $\frac{5}{4}$, which it has when $x = \frac{1}{2}$. To find such values (when practicable) by Algebra, put the given function (using x for the variable quantity) $= m$, and solve this equation: by this means x will be expressed in terms of m, and it will be easy to see what will be the greatest or least values allowable for m, so that x may be a *possible* quantity.

Ex. 1. Let $3x^2 - 3x + 2 = m$, or $3x^2 - 3x + (2 - m) = 0$: now here (129) in order that x may be *possible*, we must have 9 not $< 12(2 - m)$, and therefore m not $< \frac{5}{4}$, which is consequently its *minimum* value, in which case $x = \frac{1}{2}$.

Ex. 2. Let $\dfrac{x^2 + x + 1}{x^2 + 1} = m$, or $(1 - m)x^2 + x + (1 - m) = 0$: and here, in order that x may be possible, we must have $1 > 4(1 - m)^2$, or $m^2 - 2m + \frac{3}{4} < 0$, which [75] is the case for all values of m between $\frac{1}{2}$ and $\frac{3}{2}$, the roots of the equation $m^2 - 2m + \frac{3}{4} = 0$: hence $\frac{1}{2}$ and $\frac{3}{2}$ are the *Limits* of the possible values of the given function of x.

Ex. 16.

Find the *maximum* or *minimum* value, determining which,

1. Of $x^2 - 6x + 16$, $x^2 + 3x + 4$, $(x - 3)(4 - x)$, $(a - x)(x - b)$.

2. Of $\dfrac{x - 4}{(x - 1)^2}$, $ax + \dfrac{b}{x}$, $\dfrac{a^2 + x^2}{a^2 - x^2}$, $\dfrac{x^2 - x - 6}{x^2}$, $\dfrac{(a + x)(x - b)}{x^2}$.

3. Divide a number a into two parts so that (i) their product, (ii) the sum of their squares, (iii) the sum of the quotients of one by the other, may be a *maximum* or *minimum*, determining which.

4. Find what values are possible for the expressions
$\dfrac{x^2 - x + 1}{x^2 + x + 1}$, $\dfrac{x^2 - 3x + 4}{x^2 + 3x + 4}$, $\dfrac{(x + 1)(x + 2)}{x(x + 3)}$, $\dfrac{(a + x)(b + x)}{x}$.

CHAPTER V.

PROGRESSIONS, PROPORTIONS, AND COMBINATIONS.

77. To explain the negative root in (143 Ex. 3) write $-n$ for n, and we have $S=\{2a-(n+1)d\}\times -\frac{1}{2}n=\{2(d-a)+(n-1)d\}\frac{1}{2}n$, which is the sum of n terms of the *same* AR. series, but with $d-a$ instead of a for the first term. Thus 24 is the sum, not only of $3+5+7+9$, but of $-1+1+3+5+7+9$, where the first term is $d-a=2-3=-1$, and *last* term 9, the same as before.

It is easily shewn that in all such cases the last terms will be the same; for since we have $n^2+\left(\frac{2a}{d}-1\right)n=\frac{2S}{d}$, it follows by (130) that, if n, $-n'$, be the two values of n, then $-n+n'=\frac{2a}{d}-1$;

$\therefore\ n'd=2a+(n-1)d$, and $l'=(d-a)+(n'-1)d=-a+n'd=a+(n-1)d=l$.

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78. (161) This is the definition of Proportion which is employed in Geometry, and which is thus shewn to be identical with that given in Arithmetic and Algebra. The reason why we are obliged to have recourse to another test of proportionality, although a less simple and obvious one, in geometrical reasoning, is that there is no geometrical mode of expressing a ratio, nor therefore of proving the equality of two ratios. For a ratio, being a mere number, (though the things compared may be geometrical quantities, lines, areas, or solids,) cannot be represented by a geometrical quantity, a line of certain length for instance, except by departing from pure Geometry, (in which a line always stands for a line and nothing else,) and using the *line* in a symbolical sense, just in the same way as we might take a *letter* x to represent the same number in Algebra. Accordingly we find that the definition of Ratio in Geometry, viz. 'the relation of quantities of the same kind with respect to magnitude,' is so vague as not to deserve the name of a definition: nor is any proposition founded upon it, the reasonings of Euclid being confined to ratios when considered in connexion with each other, in the shape of a proportion.

79. If $a:b::c:d$, and a be the *greatest*, shew that d is the *least*, and $a+d>b+c$.

Since $\frac{a}{b} = \frac{c}{d}$, and $a > b$, $\therefore c > d$; also, since $\frac{a}{c} = \frac{b}{d}$, and $a > c$, $\therefore b > d$; and a, the *greatest*, is, of course, $> d$; $\therefore d$ is the *least*: again $\frac{a-b}{a} = \frac{c-d}{c}$, and $a > c$, $\therefore a - b > c - d$, or $a + d > b + c$.

This appears from (165 Ex. 3); since $(a+d) = (b+c) + \frac{(a-b)(a-c)}{a}$, which last quantity is *positive*, since $(a-b)$ and $(a-c)$ are *positive*.

80. (165) More generally, if four quantities are proportionals, then any proportion whatever will be true, in which the terms are all *homogeneous*, and those of the first ratio the *same functions* of the 1st and 2nd quantities that those of the second ratio are of the 3rd and 4th, or else, those of the first ratio the *same functions* of the 1st and 3rd quantities, that those of the second ratio are of the 2nd and 4th: and the proportions thus formed may yet further be changed by *alternation*.

Thus, if $a : b :: c : d$, then $a^2 - ab + b^2 : a^2 + 3b^2 :: c^2 - cd + d^2 : c^2 + 3d^2$,

or, *alternately*, $a^2 - ab + b^2 : c^2 - cd + d^2 :: a^2 + 3b^2 : c^2 + 3d^2$;

and, in like manner, $a^2 - ac + c^2 : a^2 + 3c^2 :: b^2 - bd + d^2 : b^2 + 3d^2$,

or, *alternately*, $a^2 - ac + c^2 : b^2 - bd + d^2 :: a^2 + 3c^2 : b^2 + 3d^2$.

As the above is an important proposition, the student's attention should be again directed to the Proof of it in [27].

81. In like manner by [28], if $a : b :: c : d :: e : f$, we have
$a : b :: c + e : d + f :: ma - ne : mb - nf :: a - c - e : b - d - f$, &c.
$a^2 : b^2 :: a^2 - e^2 : b^2 - f^2 :: (a+c)^2 : (b+d)^2 :: (a - mc + ne)^2 : (b - md + nf)^2$, &c.
$$a^3 + c^3 : ace + e^3 :: b^3 + d^3 : bdf + f^3, \text{ \&c.}$$

82. Single and Double Rule of Three sums are solved upon the principles of Variation and Proportion.

In *Single* Rule of Three, we have given corresponding values of two things which vary as one another, and have then to find the change produced in one when a given change is made in the other.

Thus, in the question, 'If 57 cwt. cost £216, what will 95 cwt. cost?' we have given 57 cwt. and £216, corresponding quantities of *weight* and *price*; and have to find to what £216 will be changed, when 57 cwt. is changed to 95 cwt. Now weight $\propto$ price *directly*;

$$\therefore 57 : 95 :: 216 : \text{the answer} = \tfrac{1}{57} \times 95 \times 216 = 360.$$

Again, in the question, 'How many men would do in 168 days a piece of work which 108 men can do in 266 days?' we have given 266 days and 108 men, corresponding quantities of *time* and *labour;* and have to find to what 108 men must be changed, when 266 days are changed to 168 days. Now the time for a given work increases in the same proportion as the No. of men employed decreases, and *vice versâ*, *i.e.* time $\propto$ No. of men *inversely;*

$$\therefore \tfrac{1}{266} : \tfrac{1}{168} :: 108 : \text{the answer} = \tfrac{1}{168} \times 108 \div \tfrac{1}{266} = 171.$$

From (164) we see that we may simplify a Rule of Three sum by dividing the 1st and 2nd, or 1st and 3rd terms, by any common factor which they may contain.

83. In *Double* Rule of Three, we have given one thing, which varies separately as *each* of several others, if the rest be constant, and, therefore, varies as their product, when all are changeable (171). Having given then one value of the former or single quantity, and corresponding values of the others, we have to find to what the first will be changed, when the latter are changed to others also given.

Thus, in the question, 'If 12 horses plough 11 acres in 5 days, how many horses would plough 33 acres in 18 days?' the single term is *horses*, and the others are *acres* and *days*. Now if the No. of days were constant, the No. of horses would vary as that of acres *directly;* again, if the No. of acres were constant, the No. of horses would vary as that of days *inversely;* hence the No. of horses $\propto \dfrac{\text{No. of acres}}{\text{No. of days}}$, when both change their values;

$$\therefore \tfrac{11}{5} : \tfrac{33}{18} :: 12 : \text{the answer} = \tfrac{33}{18} \times 12 \div \tfrac{11}{5} = 10.$$

84. It follows easily from [27] that if $A \propto B$, then any homogeneous functions whatever of A and B vary as each other.

For if $A \propto B$, or $\dfrac{A}{B} = m = \dfrac{m}{1}$, then any fraction whatever, formed by means of A and B, with num^r and den^r homogeneous, is equal to a similar fraction, with m and 1 in the place of A and B, and therefore is *constant* in value; hence the num^r $\propto$ the den^r.

Thus if $A \propto B$, then $\dfrac{A^3 + 3A^2B}{2AB^2 - B^3} = \dfrac{m^3 + 3m^2}{2m - 1}$ = a constant, and $\therefore A^3 + 3A^2B \propto 2AB^2 - B^3$; an expression of which kind (since it

may be immediately turned into an equation, by means of a constant factor) it will be convenient to call an *equation of variation.*

85. If $A \propto B \propto C$, then an equation of variation obtains between any two *homogeneous* functions whatever of A, B, C.

For we may express each of A and B in terms of C by a constant factor, as $A = mC$, $B = nC$: making these substitutions in any fraction, whose terms are *homogeneous* functions of A, B, C, we shall be able to strike out C altogether, and the fraction will be seen to have a *constant* value; hence the num^r $\propto$ the den^r.

Thus $\dfrac{A^3+2ABC}{B^2C-C^3} = \dfrac{m^3+2mn}{n^2-1}$ = a constant; $\therefore A^3+2ABC \propto B^2C-C^3$.

The same proof may plainly be extended to any number of quantities, A, B, C, D, &c.

86. Conversely, if an equation of variation obtain between two homogeneous functions of A and B, then $A \propto B$.

For putting in the given equation $A = xB$ (without knowing as yet whether x is constant or variable) we shall find that B may be struck out, and there will remain an equation, with constant coeff^s, for determining x; hence x will be constant, and $A = xB \propto B$.

Thus if $A^2 - 2AB \propto 3AB - B^2 = m(3AB - B^2)$, putting xB for A, we have $x^2 - 2x = m(3x - 1)$, where, since m is constant, the values of x will be constant, and $\therefore A = xB \propto B$.

Hence also it follows now by [84] that if an equation of variation obtain between any two hom. functions of A and B, there will obtain one also between any other two hom. functions whatever.

87. If *two* equations of variation be given between hom. functions of *three* quantities, A, B, C, then $A \propto B \propto C$.

For putting $A = xC$, $B = yC$, in the given equations, we shall be able to strike out C as before, and there will result two equations with constant coeff^s for determining x and y, which will therefore have constant values; hence $A \propto C \propto B$.

In like manner it may be shewn that if $n-1$ equations of variation be given between hom. functions of n quantities, A, B, C, D, &c., then $A \propto B \propto C \propto D$, &c.; and hence also an equation of variation will obtain between any other two hom. functions whatever of these quantities.

Ex. If $AB \propto C^2$ and $A^2 \propto BC$, then $A \propto B \propto C$, and $A^2 + C^2 \propto (A + C)B$.

For let $A = xC$, $B = yC$; then, if $AB = mC^2$ and $A^2 = nBC$, we have (i) $xy = m$, (ii) $x^2 = ny$; from which equations x and y may be found in terms of m and n, and are therefore *constant;* hence

$$A \propto C \propto B, \text{ and } \frac{A^2 + C^2}{(A + C) B} = \frac{x^2 + 1}{(x + 1) y} = \text{a constant};$$

$$\therefore A^2 + C^2 \propto (A + C) B.$$

Ex. 17.

1. If $A \propto B$, then $A^2 \pm B^2 \propto AB$, and $A^2B \pm AB^2 \propto (A \pm B)^3$.
2. If $A + B \propto C$, and $B + C \propto A$, then $A + C \propto B$, and $A^2 \propto BC$.
3. If $A \propto B + C$, $D^2 \propto B^2 + C^2$, and $AD \propto BC$,
   then $A \propto B \propto C \propto D$, and $A^2 + C^2 \propto B^2 + D^2 \propto AC + BD$.
4. If $A \propto B$, $AC \propto D^2$, and $B^2 \propto CD$,
   then $(A + B + C)^4 \propto ABCD \propto A^4 + B^4 + C^4$.

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88. *The No. of Var*^{ns} *of* n *different letters, taken* r *together, when each may enter* 1, 2, 3, *&c. or* r *times in each Var*^{n}, is n^r.

For, when taken *two* together, a may stand before itself, or any other of the n letters, as *aa, ab, ac,* &c.; that is, there will be n Var^{ns} of this kind, in which a stands first: similarly, of b, c, &c.: on the whole, therefore, there will be n^2 Var^{ns} of this kind, when the letters are taken *two* together.

Again, before each of these a may be set, and so there will be n^2 Var^{ns} of this kind, when the letters are taken *three* together, in which a stands first; similarly of b, c, &c.; that is, on the whole, there will be n^3 such Var^{ns}: and so on.

Cor. The sum of such Var^{ns} of n letters, taken 1, 2, 3, &c. r together, $= n + n^2 + \text{\&c. } n^r = n \cdot \dfrac{n^r - 1}{n - 1}$.

89. *To find what No.* r *out of* n *things must be taken together, so that the No. of Combinations formed may be the greatest possible.*

Since C_r is obtained by multiplying C_{r-1} by $\dfrac{n - r + 1}{r}$ or $\dfrac{n+1}{r} - 1$, the quantities C_1, C_2, &c. will increase continually, each term upon the preceding, so long as this factor > 1; hence C_r will be the greatest for the greatest value of r which allows of this, or of $n + 1 > 2r$, that is, when r is the integer next $< \frac{1}{2}(n + 1)$.

If n be *even*, $r=\frac{1}{2}n$: if n be *odd*, and $\therefore \frac{1}{2}(n+1)$ an integer, $r=\frac{1}{2}(n+1)-1=\frac{1}{2}(n-1)$: but, in this case, since (178) $C_r=C_{n-r}$, the n° taken $\frac{1}{2}(n-1)$ together = the n° taken $n-\frac{1}{2}(n-1)$ or $\frac{1}{2}(n+1)$ together, and $\therefore r$ may be either $\frac{1}{2}(n-1)$ or $\frac{1}{2}(n+1)$.

Ex. If $n=6$, $r=\frac{1}{2}n=3$, $C_3=20$; if $n=7$, $r=\frac{1}{2}(n\pm 1)=4$ or 3, $C_3=35=C_4$.

90. *If there be several sets, containing* n_1, n_2, n_3, *&c. things respectively, the n° of Comb^ns formed by taking one out of each set, is* $n_1 . n_2 . n_3$. *&c.*

For let there be two such sets, as a_1, a_2, &c., b_1, b_2, &c.; then, since each of the n_1 things in the first may be combined with each of the n_2 in the second, the n° of Comb^ns of *two* things, one out of each set, will be $n_1 . n_2$. Again, if there be a third set of n_3 things, c_1, c_2, &c., each of the former $n_1 . n_2$ Comb^ns may be taken with each of these n_3 things, and will thus form $n_1 . n_2 . n_3$ Comb^ns of *three* things, one out of each set; and so on.

Cor. If there be the same n°, n, in each set, and r such sets, the n° of Comb^ns thus formed is n^r.

Ex. 18.

1. Find the greatest n° of Comb^ns that can be made out of 10 things; and the whole n° when taken 1, 2, 3, &c. 10 together.

2. How many letters of the word *holidays* should be taken together so as to produce the greatest n° of different words? What is the difference between that n° of words and the greatest which can be produced by taking letters of the word *universal*?

3. A person wishes to make up as many *different* dinner parties as he can, out of 24 friends; how many should he invite at a time?

4. Six yachts are to be so arranged in a regatta, that there may be the greatest possible n° of different matches; how many then must sail in each match, and how many matches would there be?

5. In what numbers should 20 men be combined so as to form the greatest possible n° of different companies? In how many of these would the same man be found?

6. There are four regular polyhedrons marked in the manner of dice, and the numbers on their faces are 3, 6, 8, 12, respectively: how many different throws can possibly be made, taking all of them together?

CHAPTER VI.

BINOMIAL, AND MULTINOMIAL THEOREMS.

VARIOUS proofs have been given of the Binomial Theorem. Besides that given in the Text, the following is worthy of notice, as being a complete proof for all values of the index.

91. *To prove the Binomial Theorem for all values of the Index.*

If n be a *positive integer*,

$$\frac{(1+x)^n-1}{(1+x)-1} = (1+x)^{n-1} + (1+x)^{n-2} + \&c. + (1+x)^2 + (1+x) + 1;$$

$$\therefore (1+x)^n = 1 + x\{(1+x)^{n-1} + (1+x)^{n-2} + \&c. + (1+x) + 1\},$$
$$= 1 + xp \text{ suppose};$$

Similarly each of the quantities $(1+x)^{n-1}$, $(1+x)^{n-2}$, &c. may be expressed in the form $1+xp_1$, $1+xp_2$, so that

$$(1+x)^n = 1 + x\{(1+xp_1) + (1+xp_2) + \&c.\} = 1 + x\{n + x(p_1+p_2+\&c.)\}$$
$$= 1 + nx + \text{terms in } x^2,\ x^3, \&c.$$

Again, if $n=-m$, a *negative integer*,

$$\frac{(1+x)^{-m}-1}{(1+x)^{-1}-1} = (1+x)^{-(m-1)} + (1+x)^{-(m-2)} + \&c. + (1+x)^{-1} + 1,$$

$$\text{and} \quad (1+x)^{-1} - 1 = \frac{1}{1+x} - 1 = -\frac{x}{1+x} = -x(1+x)^{-1};$$

$$\therefore (1+x)^{-m} - 1 = -x\{(1+x)^{-m} + (1+x)^{-(m-1)} + \&c. + (1+x)^{-2} + (1+x)^{-1}\},$$

and, as before, $(1+x)^{-m} = 1 - x\{m - \&c.\} = 1 - mx +$ terms in x^2, x^3, &c.

Lastly, if $n = \frac{p}{q}$, a *fraction*, p and q being + or − integers, assume $(1+x)^{\frac{p}{q}} = 1 + Ax + Bx^2 + \&c.$, whence $(1+x)^p = (1 + Ax + \&c.)^q$;

$\therefore$, by the former cases, $1 + px + \&c. = 1 + q(Ax + \&c.) + \&c.$;

whence, equating coefficients of x,

$$p = qA, \text{ and } A = \frac{p}{q}, \text{ or } (1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}x + \text{terms in } x^2,\ x^3, \&c.$$

We may assume them, for all values of the index,

$$(1+x)^n = 1 + Ax + Bx^2 + Cx^3 + \&c., \text{ where } A = \text{the index, } n.$$

Hence $(1+x+z)^n = 1 + A(x+z) + B(x+z)^2 + C(x+z)^3 + \&c.$;

$$\text{but } (1+x+z)^n = \left\{(1+x)\left(1+\frac{z}{1+x}\right)\right\}^n = (1+x)^n\left(1+\frac{z}{1+x}\right)^n$$

$$= (1+x)^n \left\{1 + A\left(\frac{z}{1+x}\right) + B\left(\frac{z}{1+x}\right)^2 + C\left(\frac{z}{1+x}\right)^3 + \&c.\right\}:$$

$\therefore$ equating coeff^{s} of z in these identical expansions of $(1+x+z)^n$, we have $\quad A + 2Bx + 3Cx^2 + \&c. = A(1+x)^{n-1}$

$$= A\{1 + A'x + B'x^2 + C'x^3 + \&c.\},$$

where A', B', &c. correspond in the expansion of $(1+x)^{n-1}$ to A, B, &c. in that of $(1+x)^n$, and may be formed from A, B, &c., by writing in them $n-1$ for n; thus, since $A = n$, $A' = n-1$: equating now coefficients of x in these expressions, we have

$$2B = AA', \quad 3C = AB', \quad \&c.,$$

whence $B = \dfrac{AA'}{2} = \dfrac{n(n-1)}{1.2}$, $\qquad$ and $\therefore$ $B' = \dfrac{(n-1)(n-2)}{1.2}$,

$$C = \frac{AB'}{3} = \frac{n(n-1)(n-2)}{1.2.3}, \quad \ldots\ldots \quad C' = \frac{(n-1)(n-2)(n-3)}{1.2.3},$$

$$\&c. = \&c. \qquad\qquad \&c. = \&c.$$

and $\therefore$ $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{1.2}x^2 + \dfrac{n(n-1)(n-2)}{1.2.3}x^3 + \&c.$

Hence $(a+x)^n = a^n\left(1+\dfrac{x}{a}\right)^n = a^n\left\{1 + n\left(\dfrac{x}{a}\right) + \dfrac{n(n-1)}{1.2}\left(\dfrac{x}{a}\right)^2 + \&c.\right\}$

$$= a^n + na^{n-1}x + \tfrac{1}{2}n(n-1)a^{n-2}x^2 + \&c.,$$

where coeff. of $a^{n-r}x^r = \dfrac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}$ =, as in (182), $\dfrac{\lfloor n}{\lfloor r \lfloor n-r}$.

92. If $(1+x)^{n-1} = 1 + B_1x + B_2x^2 + \&c. + B_rx^r + \&c.$, then, multiplying both sides of this equation by $1+x$, we get

$$\begin{aligned}(1+x)^n = 1 &+ B_1x + B_2x^2 + \&c. + B_rx^r + \&c.\\ &+ \quad x + B_1x^2 + \&c. + B_{r-1}x^r + \&c.\\ \hline = 1 &+ C_1x + C_2x^2 + \&c. + C_rx^r + \&c.,\end{aligned}$$

where $C_1 = 1 + B_1$, $C_2 = B_1 + B_2$, &c., and, generally, $C_r = B_{r-1} + B_r$.

These results are proved independently of the *form* of the series for $(1+x)^{n-1}$ and $(1+x)^n$: but assuming the form to be known as above, we may also prove them easily by actual addition, whatever be the value of n.

For $C_r = \dfrac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}}$; $\therefore$, writing $n-1$ for n, $B_r = \dfrac{\lfloor\underline{n-1}}{\lfloor\underline{r}\,\lfloor\underline{n-r-1}}$,

and $\therefore$, writing $r-1$ for r, $B_{r-1} = \dfrac{\lfloor\underline{n-1}}{\lfloor\underline{r-1}\,\lfloor\underline{n-r}}$;

$$\therefore B_{r-1} + B_r = \frac{\lfloor\underline{n-1}}{\lfloor\underline{r-1}\,\lfloor\underline{n-r-1}}\left\{\frac{1}{r} + \frac{1}{n-r}\right\}$$

$$= \frac{\lfloor\underline{n-1}}{\lfloor\underline{r-1}\,\lfloor\underline{n-r-1}}\left\{\frac{n}{r(n-r)}\right\} = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}} = C_r.$$

COR. 1. Since $C_r = B_{r-1} + B_r$, or any coeff. in the expansion of $(1+x)^n$ = the sum of two coeff^s in that of $(1+x)^{n-1}$, it follows that, if *all* the coeff^s in the latter expansion be *integers*, so also will they be in the former: but all the coeff^s *are* integers in the expansions of the *second, third,* &c. powers; therefore in that of the *fourth,* &c.; and so, generally, *all the coeff^s are integers in the expansion of any positive integral power of* $1+x$.

COR. 2. Since, then, if r be any positive integer, C_r is also an integer, it follows that the num^r $n(n-1)\dots(n-r+1)$ is divisible without rem^r by the den^r $1.2\dots r$; or, since n, $n-1$, &c. $n-r+1$, are consecutive integers, r in number, and we may take the first of them, n, to stand for any integer whatever, it may be said that *the product of any* r *consecutive integers is divisible by* 1.2 ... r.

COR. 3. From the result $C_r = B_{r-1} + B_r$, we may easily form the following table, which goes by the name of *Pascal's Triangle,* and in which the horizontal line gives the coeff^s of the successive integral powers of $1+x$, and the terms in each horizontal line are formed from those in the next preceding.

Values of Index.	
0	1
1	1 + 1
2	1 + 2 + 1
3	1 + 3 + 3 + 1
4	1 + 4 + 6 + 4 + 1
5	1 + 5 + 10 + 10 + 5 + 1
6	1 + 6 + 15 + 20 + 15 + 6 + 1

93. By attending to the Law of *homogeneous* quantities referred to in (107 Ex. 2), we may facilitate the expansion of many binomials: thus $(a-2x)^{-\frac{3}{2}}$ is homogeneous and of $-\frac{3}{2}$ dimensions; if then we expand $(1-2x)^{-\frac{3}{2}}$, we need only introduce into *each* term such a power of a, as will make it of $-\frac{3}{2}$ dimensions.

Ex. 1. $(1+x-x^2)^{\frac{3}{2}} = 1 + \frac{3}{2}(x-x^2) + \frac{3}{8}(x-x^2)^2 - \frac{1}{16}(x-x^2)^3 - \text{\&c.}$
$=1+\frac{3}{2}(x-x^2)+\frac{3}{8}(x^2-2x^3+x^4)-\frac{1}{16}(x^3-3x^4+\text{\&c.})+\text{\&c.}=1+\frac{3}{2}x-\frac{9}{8}x^2-\frac{13}{16}x^3+\text{\&c.}$

Ex. 2. $(a^2+ax-x^2)^{\frac{3}{2}} = a^3 + \frac{3}{2}a^2x - \frac{9}{8}ax^2 - \frac{13}{16}x^3 + \frac{75}{128}a^{-1}x^4 + \text{\&c.},$
which, since $(a^2+ax-x^2)^{\frac{3}{2}}$ is homogeneous and of 3 dimensions, is derived from Ex. 1 by merely introducing into each term such a power of a as will make it of 3 dimensions:

Ex. 19.

1. $(a-2x)^{-\frac{1}{2}}$. 2. $(a^2-3ax)^{-\frac{1}{3}}$. 3. $\sqrt[4]{(a^3+4a^2x)}$. 4. $\{a+\sqrt{(ax)}\}^{\frac{1}{2}}$.
5. $(1-x-x^2)^{-1}$. 6. $(1-x-x^2)^{-2}$. 7. $(1+x-x^2)^{\frac{1}{2}}$.
8. $(1-3x+x^2)^{\frac{1}{3}}$. 9. $(a^2-ax-x^2)^{-\frac{2}{3}}$. 10. $(a^2+2ax-x^2)^{-2}$.

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94. The student will now do well to exercise himself in setting down the *general* term of any expansion, and to notice the different forms which the general expression for C_r assumes for different values of n.

(i) If n be a *positive integer*, the ordinary form is to be used, the factors in the num^r being r in n°. (observe, *one for each in the den^r*) and *diminishing* regularly by unity; but in the expansion of $(1-x)^n$ or $\{1+(-x)\}^n$, the general term will be $C_r(-x)^r$ or $C_r.(-1)^r.x^r=(-1)^r.C_rx^r$, where the quantity $(-1)^r$ will be either $+1$ or -1, according as r is even or odd, that is, according as $r+1$, the n°. of the term, is odd or even. The factor $(-1)^r$ will therefore merely determine the *sign* of the term, without affecting its numerical value.

Ex. 1. In $(1-x)^9$ the general term is

$$(-1)^r.\frac{9.8\ldots\{9-(r-1)\}}{1.2\ldots r}x^r=(-1)^r.\frac{9.8\ldots(10-r)}{1.2\ldots r}x^r.$$

(ii) If n be a *negative* integer, C_r becomes

$$\frac{-n(-n-1)\ldots\{-n-(r-1)\}}{1.2\ldots r}=(-1)^r.\frac{n(n+1)\ldots(n+r-1)}{1.2\ldots r};$$

and here the factors of the num^r go on *increasing* by unity, until we come to the last of them, $n+r-1$.

If the binomial be $(1-x)^{-n}$, then, as before, the general term is

$$(-1)^r.C_r(-x)^r=(-1)^r.(-1)^r.\frac{n(n+1)\ldots(n+r-1)}{1.2\ldots r}x^r=\frac{n(n+1)\ldots(n+r-1)}{1.2\ldots r}x^r,$$

since $(-1)^r.(-1)^r$ or $(-1)^{2r}$, being an *even* power of -1, is always $+1$.

Ex. 2. In $(1+x)^{-3}$ the general term is

$$(-1)^r \cdot \frac{3.4.5 \ldots (3+r-1)}{1.2.3 \ldots r} x^r = (-1)^r \cdot \frac{(r+2)(r+1)}{1.2} x^r.$$

Notice then that (n being an *integer*) all the coeff^s of $(1+x)^n$ or $(1-x)^{-n}$ are positive, while in $(1-x)^n$ and $(1+x)^{-n}$ they are alternately positive and negative: in the former case the general term is positive, in the latter it involves the factor $(-1)^r$.

(iii) The same Laws with regard to signs obtain also in the case of a *fractional* index, only here, as in (184 Ex. 5), C_r may be written

$$\frac{p(p-q) \ldots \{p-(r-1)q\}}{1.2 \ldots r.q^r} \text{ for } (1+x)^{\frac{p}{q}},$$

$$\text{and } \frac{p(p+q) \ldots \{p+(r-1)q\}}{1.2 \ldots r.q^r} \text{ for } (1-x)^{-\frac{p}{q}},$$

and the same with the factor $(-1)^r$, for $(1-x)^{\frac{p}{q}}$ and $(1+x)^{-\frac{p}{q}}$.

Ex. 3. In $(1+x)^{-\frac{3}{2}}$, the general term is

$$(-1)^r \cdot \frac{3.5.7 \ldots \{3+(r-1)2\}}{1.2.3 \ldots r.2^r} x^r = (-1)^r \cdot \frac{3.5.7 \ldots (2r+1)}{1.2.3 \ldots r.2^r} x^r.$$

Ex. 4. In $(1+x)^{\frac{7}{3}}$, it is $\dfrac{7.4.1.(-2).(-5) \ldots \{7-(r-1)3\}}{1.2.3.4.5 \ldots r.3^r} x^r$.

Now here we see that negative factors enter in the num^r, and it is plain that the same will occur in *all* cases where the index is a *positive fraction*. To avoid the awkwardness of writing these, we may change the signs of them all, when they will of course go on increasing instead of, as before, decreasing, (thus in Ex. 4, we may write the num^r 7.4.1.2.5...), only we must then prefix, as a factor, some power of (-1), according to the original n°. of negative factors. Now, if the factors in the num^r had been originally *all* negative, the factor in question would have been, as in Ex. 2, $(-1)^r$; but if any *even* n°. of these negative factors be exchanged for positive, the sign of the whole product will not be altered, and may therefore still be expressed by $(-1)^r$; whereas if any *odd* n°. of them be changed, the sign of the whole product *will* be altered, and may be expressed by $-(-1)^r$ or $(-1)^{r+1}$. In such a case then we shall have to prefix $(-1)^r$ or $(-1)^{r+1}$, according as the n°. of positive factors is even or odd.

This modification, however, of the general term will, of course, apply only to such terms of the expansion as involve negative factors;

thus, in Ex. 4, we may write the term, $(-1)^{r+1}\dfrac{7.4.1.2.5\ldots(3r-10)}{1.2.3.4.5\ldots r.3^r}x^r$; but this is only true after the *third* term, and therefore is not, strictly speaking, the expression for the *general* term, as the form originally written was. However, we shall still call it the general term, with the understanding that it expresses any term of the series, as soon as the negative factors begin to enter the num^r, but not before: and of this we shall be reminded by the peculiarity of the factors in the num^r first decreasing and then increasing.

The above will apply to the case of $(1+x)^{\frac{p}{q}}$; but in that of $(1-x)^{\frac{p}{q}}$, the general term will of itself by (i) involve $(-1)^r$; and this factor, combined with the above, will become $(-1)^{2r}$ or $(-1)^{2r+1}$, that is, $+1$ or -1, according as the n°. of positive factors is even or odd. Hence it follows that, in this case, as soon as a negative factor enters the coeff^s the terms of the series will become either *all positive* or *all negative*, according as the general term takes the factor $+1$ or -1, that is, according as the n°. of positive factors is *even* or *odd*.

Ex. 5. In $(1-x)^{\frac{5}{2}}$ the general term is $-\dfrac{5.3.1.1.3\ldots(2r-7)}{1.2.3.4.5\ldots r.2^r}x^r$, where the last factor in the num^r was originally $5-(r-1)\,2$, which, with sign changed, becomes $-5+(r-1)\,2=2r-7$; and the sign $-$ is prefixed because the n°. of positive factors is *odd*, and *all* the terms after the fourth will be *negative*.

Ex. 6. In $(a-3x)^{\frac{2}{3}}=a^{\frac{2}{3}}\left\{1-\dfrac{3x}{a}\right\}^{\frac{2}{3}}$, the general term will be

$$-a^{\frac{2}{3}}\left\{\frac{2.1.4.7\ldots(3r-5)}{1.2.3.4\ldots r.3^r}\cdot\left(\frac{3x}{a}\right)^r\right\}=-a^{\frac{2}{3}}\left\{\frac{2.1.4\ldots(3r-5)}{1.2.3\ldots r}\,\frac{x^r}{a^r}\right\}.$$

Ex. 20.

Find the general terms in

1. $(1+x)^8$. 2. $(1+x)^{12}$. 3. $(a-x)^9$. 4. $(3x+y)^7$. 5. $(1+x)^{-1}$.
6. $(1+3x)^{-2}$. 7. $(1-2x)^{-3}$. 8. $(1-x)^{-\frac{3}{2}}$. 9. $(1-x)^{-\frac{1}{3}}$.
10. $(1-x^2)^{-\frac{1}{2}}$. 11. $(2-x)^{-2}$. 12. $(a+bx)^{-1}$.
13. $(a^{-\frac{1}{3}}+b^{-\frac{1}{3}})^{-3}$. 14. $(1+2x)^{\frac{1}{2}}$. 15. $(1-3x)^{\frac{2}{3}}$.
16. $(a^2-x^2)^{\frac{2}{3}}$. 17. $(a^2-x^2)^{-\frac{3}{4}}$. 18. $(ax-x^2)^{-\frac{1}{3}}$.

95. *To find the greatest term of* $(1+x)^n$.

The $(r+1)^{\text{th}}$ term is obtained from the r^{th} by multiplying it by the factor $\dfrac{n-r+1}{r}x$, or $\left\{\dfrac{n+1}{r}-1\right\}x$: now as r increases, $\dfrac{n+1}{r}$ diminishes; yet as long as the above factor >1, each term continues to increase upon the one preceding, and the greatest term will correspond to the first value of r, which makes it <1, for then the $(r+1)^{\text{th}}$ term will be less than the r^{th}, and so, *a fortiori*, will all the following terms. Hence the r^{th} term will be the greatest for the first or lowest value of r which makes $\left(\dfrac{n+1}{r}-1\right)x<1$, or $\dfrac{n+1}{r}x<1+x$, or $r>(n+1)\dfrac{x}{1+x}$.

If $(n+1)\dfrac{x}{1+x}$ *be* an integer, this may be taken for r; and then the r^{th} and $(r+1)^{\text{th}}$ terms will be equal.

In the case of $(1-x)^n$, since we only consider here the *magnitude*, not the *sign*, of the term, the result would have been the same.

But if the index be negative $=-n$, then, as regards *magnitude*, the $(r+1)^{\text{th}}$ term is formed from the r^{th} by multiplying it by $\dfrac{n+r-1}{r}x$; and now we must take the lowest value of r which makes this factor <1, whence r is the integer next $>(n-1)\dfrac{x}{1-x}$.

Ex. 1. $(1+\frac{5}{2})^{\frac{3}{2}}$: Here r is the integer next $>(\frac{3}{2}+1)\dfrac{\frac{5}{2}}{1+\frac{5}{2}}$ or $1\frac{11}{14}$; therefore the greatest term is the *second* term.

Ex. 2. $(1+\frac{5}{2})^{-\frac{3}{2}}$: Here r is the integer next $>(\frac{3}{2}-1)\dfrac{\frac{5}{2}}{1-\frac{5}{2}}$, which, being negative, shews that we cannot here determine the greatest term, the series, in fact, increasing continually.

Ex. 21.

Find the number and magnitude of the greatest term in

1. $(1+x)^6$, when $x=\frac{1}{2}$.
2. $(1-x)^{-6}$, when $x=\frac{1}{2}$.
3. $(1+x)^{\frac{3}{2}}$, when $x=\frac{2}{3}$.
4. $(1+x)^{\frac{9}{5}}$, when $x=3$.
5. $(1+x)^{-4}$, when $x=\frac{2}{5}$.
6. $(1-x)^{-\frac{4}{3}}$, when $x=\frac{12}{13}$.

96. *To find the No. of homogeneous products of* r *dimensions, that can be made out of* m *things*, a, b, c, *&c. and their powers.*

By common division, we have $\frac{1}{1-ax} \times \frac{1}{1-bx} \times \frac{1}{1-cx} \times$ &c.
$= (1 + ax + a^2x^2 + \text{&c.}) \times (1 + bx + b^2x^2 + \text{&c.}) \times (1 + cx + c^2x^2 + \text{&c.}) \times \text{&c.}$
$= 1 + (a + b + c + \text{&c.})\, x + (a^2 + ab + ac + b^2 + bc + c^2 + \text{&c.})\, x^2 + \text{&c.}$
$= 1 + S_1x + S_2x^2 + S_3x^3 + \text{&c.}$, suppose, where S_1, S_2, S_3 &c. are manifestly the sums of the homogeneous products of *one*, *two*, *three*, &c. dimensions, that can be made out of a, b, c, &c.

To obtain the *number* of these products, put $a = b = c = \text{&c.} = 1$; by this means each product, of whatever dimensions, will be reduced to 1, and the value of S_1, S_2, &c. will now be merely the *number* of such products of each class. But by this substitution, the first side of the above identity becomes

$$\left(\frac{1}{1-x}\right)^m = (1-x)^{-m} = 1 + mx + \text{&c.};\ \therefore\ S_1 = m,\ S_2 = \frac{m(m+1)}{1.2},\ \text{&c}$$

$$\text{and, generally, } S_r = \frac{m(m+1)\ldots(m+r-1)}{1.2\ldots r}.$$

Cor. Hence we may obtain the n°. of terms in the expansion of $(a + b + c + \text{&c.})^n$, when n is a positive integer.

Thus the expansion of $(a+b)^n$ will contain all possible combinations of powers of a and b, as a^n, $a^{n-1}b$, $a^{n-2}b^2$, &c., such that the sum of the indices in each term may be n, that is, it will contain all the homogeneous products of n dimensions that can be made out of *two* things a and b; the n°. of terms will be therefore $\frac{2.3\ldots(2+n-1)}{1.2\ldots n} = \frac{n+1}{1}$ or $n+1$, as we found in (181).

In like manner, the number of terms in the expansion of $(a+b+c)^n$, will be the n°. of homogeneous products of n dimensions that can be made out of *three* things, and will therefore be $\frac{3.4\ldots(3+n-1)}{12\ldots n} = \frac{(n+1)(n+2)}{1.2}$: and so on.

97. We will here complete, as far as can be done algebraically, the Theory of Vanishing Fractions. Instead of finding the vanishing factor, as in [30], we may evaluate such fractions as follows.

Let $\frac{u}{v}$ represent any fraction which becomes a vanishing fraction, when $x = a$: put $a + h$ for x, and expand u and v in ascending powers of h, and suppose that we thus get $\frac{u}{v} = \frac{Ah^{\alpha} + Bh^{\beta} + \text{&c.}}{A'h^{\alpha'} + B'h^{\beta'} + \text{&c.}}$,

α being the lowest index of h in the numr and α' in the denr; then

(i) if $\alpha = \alpha'$, $\dfrac{u}{v} = \dfrac{A + Bh^{\beta-\alpha} + \&c.}{A' + B'h^{\beta'-\alpha'} + \&c.} = \dfrac{A}{A'}$, when $x = a$ or $h = 0$;

(ii) if $\alpha > \alpha'$, $\dfrac{u}{v} = \dfrac{Ah^{\alpha-\alpha'} + Bh^{\beta-\alpha'} + \&c.}{A' + B'h^{\beta'-\alpha'} + \&c.} = \dfrac{0}{A'} = 0$, when $x = a$;

(iii) if $\alpha < \alpha'$, $\dfrac{u}{v} = \dfrac{A + Bh^{\beta-\alpha} + \&c.}{A'h^{\alpha'-\alpha} + B'h^{\beta'-\alpha} + \&c.} = \dfrac{A}{0} = \infty$, when $x = a$.

Hence $\dfrac{u}{v}$ is *zero*, *finite*, or *infinite*, according as $\alpha >, =,$ or $< \alpha'$.

Ex. 1. $\dfrac{u}{v} = \dfrac{2x^3 - 3x^2 - 12x + 20}{x^3 - x^2 - 8x + 12} = \dfrac{0}{0}$, when $x = 2$: writing $2 + h$ for x, we find that $u = 9h^2 + \&c.$, $v = 5h^2 + \&c.$; therefore, since the lowest indices of h are the same in these, $\dfrac{u}{v} = \dfrac{9}{5}$, when $x = 2$.

Ex. 2. $\dfrac{\sqrt{(a^2 + ax + x^2)} - \sqrt{(a^2 - ax + x^2)}}{\sqrt{(a + x)} - \sqrt{(a - x)}} = \dfrac{0}{0}$, when $x = 0$;
put $x = 0 + h = h$, and expand in ascending powers of h;

then u becomes $a\left(1 + \dfrac{h}{2a} + \&c.\right) - a\left(1 - \dfrac{h}{2a} + \&c.\right) = h + \&c.$,

and v becomes $\sqrt{a}\left(1 + \dfrac{h}{2a} + \&c.\right) - \sqrt{a}\left(1 - \dfrac{h}{2a} + \&c.\right) = \dfrac{h}{\sqrt{a}} + \&c.$;

$$\therefore \frac{u}{v} = \sqrt{a}, \text{ when } x = 0.$$

Ex. 22.

Evaluate

1. $\dfrac{2x^3-5x^2-4x+12}{x^3 - 12x + 16}$, when $x=2$. 2. $\dfrac{\sqrt{(a + x)} - \sqrt{(2a)}}{\sqrt{(a + 2x)} - \sqrt{(3a)}}$, when $x = a$.

3. $\dfrac{x^n}{a - \sqrt[n]{(a^n - x^n)}}$, when $x=0$. 4. $\dfrac{(x^2 - a^2)^{\frac{3}{2}} + x - a}{(1 + x - a)^3 - 1}$, when $x = a$.

5. $\dfrac{a - \sqrt{(2ax - x^2)}}{a - \sqrt[3]{(2a^2x - ax^2)}}$, when $x=a$. 6. $\dfrac{\sqrt{(2a^2+2x^2)}-2\sqrt[3]{(a^2x)}}{x - a}$, when $x=a$.

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98. The following applications of the Binomial Theorem deserve notice, and will give the Student hints for similar Problems.

Ex. 1. $\sqrt[5]{3128} = \sqrt[5]{(5^5+3)} = 5\left(1 + \dfrac{3}{5^5}\right)^{\frac{1}{5}} = 5\left\{1 + \dfrac{1}{5}\cdot\dfrac{3}{5^5} - \dfrac{1.4}{1.2.5^2}\cdot\dfrac{9}{5^{10}} + \&c.\right\}$
$= 5\,(1 + .000192 - .000000073728 + \&c.) = 5.000959$ &c.

The calculation was rendered easy here by the form of the den^r;

$$\text{thus } \frac{1}{5}\cdot\frac{3}{5^5}=\frac{3}{5^6}=\frac{3.2^6}{10^6}=\frac{192}{10^6}=.000192.$$

Ex. 2. Let a be an approximate value of $\sqrt{N}$, so that $\sqrt{N}=a+x$, x being very small; then $N=a^2+2ax+x^2=a^2+(2a+x)x$:

first suppose $N=a^2+2ax$, then $x=\dfrac{N-a^2}{2a}$, and $2a+x=\dfrac{N+3a^2}{2a}$;

therefore, more nearly, $N=a^2+\dfrac{N+3a^2}{2a}x$, and $x=2a\dfrac{N-a^2}{N+3a^2}$,

and $\sqrt{N}=a+x=a\,.\,\dfrac{3N+a^2}{N+3a^2}$.

More generally, if $\sqrt[n]{N}=a+x$, it may be similarly shewn, by taking three terms of the expansion of $(a+x)^n$, that

$$\sqrt[n]{N}=a\frac{(n+1)N+(n-1)a^n}{(n-1)N+(n+1)a^n}, \text{ approximately:}$$

putting this $=a'$, and assuming $\sqrt[n]{N}=a'+x$, we shall get in the same way a yet nearer approximation to its value.

Of course if $a>\sqrt[n]{N}$, we should assume $\sqrt[n]{N}=a-x$.

Ex. 3. Let $\dfrac{a\pm x}{a}$ be any ratio in which x is small compared with a;

then $\left\{\dfrac{a\pm x}{a}\right\}^n=\left\{1\pm\dfrac{x}{a}\right\}^n=$ (approximately) $1\pm\dfrac{nx}{a}=\dfrac{a\pm nx}{a}$.

Thus $(1001)^{\frac{1}{2}}:(1000)^{\frac{1}{2}}=1001\tfrac{1}{2}:1000$ nearly $=2003:2000$.

Ex. 4. $\left(\dfrac{a+x}{a-x}\right)^n=\left(1+\dfrac{2x}{a-x}\right)^n=1+n\left(\dfrac{2x}{a-x}\right)+\dfrac{n(n-1)}{1.2}\left(\dfrac{2x}{a-x}\right)^2+\&c.,$

or $=\left(\dfrac{a-x}{a+x}\right)^{-n}=\left(1-\dfrac{2x}{a+x}\right)^{-n}=1+n\left(\dfrac{2x}{a+x}\right)+\dfrac{n(n+1)}{1.2}\left(\dfrac{2x}{a+x}\right)^2+\&c.:$

and many similar results may be obtained, by changing the form of the quantity originally given.

Ex. 5. The coeff. of the middle term or terms of $(1+x)^n$ may be expressed as follows.

(i) If n be even, there is *one* middle term, the $(\tfrac{1}{2}n+1)^{\text{th}}$, whose coeff.

$$=\frac{\lfloor n}{\lfloor \tfrac{1}{2}n \lfloor \tfrac{1}{2}n}=\frac{1.3.5\ldots(n-1)}{1.2.3\ldots\tfrac{1}{2}n}\times\frac{2.4.6\ldots n}{1.2.3\ldots\tfrac{1}{2}n}=\frac{1.3.5\ldots(n-1)}{1.2.3\ldots\tfrac{1}{2}n}2^{\frac{1}{2}n},$$

since each of the $\tfrac{1}{2}n$ factors 2, 4, 6, in the num^r $=2\times$ corresponding one in the den^r.

(ii) If n be odd, there will be *two* middle terms, the $\frac{1}{2}(n+1)^{\text{th}}$ and the $\frac{1}{2}(n+3)^{\text{th}}$; the coeff^{s} of these will be equal, and may be shewn in the same way to be $= \dfrac{1.3.5 \dots n}{1.2.3 \dots \frac{1}{2}(n+1)} \times 2^{\frac{1}{2}(n-1)}$.

Ex. 6. The sum of *all* the coeff^{s} of $(1+x)^n = 2^n$; and sum of *even* coeff^{s} = sum of *odd* coeff^{s} $= 2^{n-1}$.

For since $(1 \pm x)^n = 1 \pm C_1 x + C_2 x^2 \pm C_3 x^3 + \&c.$,

$$\therefore\ (1+1)^n = 2^n = 1 + C_1 + C_2 + C_3 + \&c. \dots\dots\dots \text{(i)},$$
$$\text{and } (1-1)^n = 0 \ = 1 - C_1 + C_2 - C_3 + \&c. \dots\dots\dots \text{(ii)};$$

hence from (i) sum of *all* coeff^{s} of $(1+x)^n = 2^n$;
and from (ii) $1 + C_2 + \&c. = C_1 + C_3 + \&c.$, or sum of even coeff^{s} = sum of odd coeff^{s}, and $\therefore$ each sum $= \frac{1}{2}(2^n) = 2^{n-1}$

Ex. 7. Since, when n is a positive integer, we have

$$(1+x)^n = 1 \ + C_1 x \ \ + C_2 x^2 \ \ + \&c. + C_2 x^{n-2} + C_1 x^{n-1} + x^n,$$
$$\text{and } (x+1)^n = x^n + C_1 x^{n-1} + C_2 x^{n-2} + \&c. + C_2 x^2 \ \ + C_1 x \ \ + 1,$$
$$\therefore \text{ by mult}^{\text{n}}\ (1+x)^{2n} = x^n + C_1 x^{n+1} + C_2 x^{n+2} + \&c.$$
$$+ C_1 x^{n-1} + C_1^2 x^n \ + \&c.$$
$$+ C_2 x^{n-2} + \&c.$$

Now the coeff. of x^n in $(1+x)^{2n}$ is (as in Ex. 5) $\dfrac{1.3.5\dots(2n-1)}{1.2.3\dots n} \times 2^n$,
and the coeff. of x^n in the above product is $1 + C_1^2 + C_2^2 + \&c.$;

$$\therefore\ 1 + n^2 + \left\{\frac{n(n-1)}{1.2}\right\}^2 + \&c. = \frac{1.3.5 \dots (2n-1)}{1.2.3 \dots n} \times 2^n.$$

By similar reasonings, equating the coeff^{s} of other powers of x, we may obtain the sums of other series, such as $1.C_1 + C_1.C_2 + C_2.C_3 + \&c.$

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99. *Multinomial Theorem:* To find the general term of $(a+b+c+d+\&c.)^m$.

For $b+c+d+\&c.$ write b'; then in the expansion of $(a+b')^m$ there will be the general term $\dfrac{m(m-1)\dots(m-r+1)}{1.2\dots r} a^{m-r} b'^r$, or, putting α for $m-r$, $\dfrac{m(m-1)\dots(\alpha+1)}{1.2\dots r} a^{\alpha} b'^r$, where r is a positive integer, whatever m may be.

Now (i) if m, and therefore α, be a positive integer, we may write this

$$\frac{m(m-1)\dots(\alpha+1)\,\alpha(\alpha-1)\dots 3.2.1}{1.2\dots\alpha \times 1.2\dots r} a^{\alpha} b'^r \text{ or } \frac{\lfloor m}{\lfloor \alpha \lfloor r} a^{\alpha} b'^r:$$

but $b'=b+c+d+\&c.=b+c'$, suppose, and in the expansion of $(b+c')^r$, (since r is a positive integer) there will be the general term $\frac{\lfloor r}{\lfloor r-s \, \lfloor s} b^{r-s}c'^s$, suppose, or $\frac{\lfloor r}{\lfloor \beta \, \lfloor s} b^\beta c'^s$, if we put β for $r-s$; combining this with the former, we have the general term of $(a+b+c')^m$, when m is a positive integer, $\frac{\lfloor m}{\lfloor a \times \lfloor \beta \times \lfloor s} a^a b^\beta c'^s$.

Proceeding thus we get the general term of $(a+b+c+\&c.)^m$ when m is a positive integer, to be

$$\frac{\lfloor m}{\lfloor a \times \lfloor \beta \times \lfloor \gamma \times \&c.} a^a b^\beta c^\gamma \ldots,$$

where the indices are all positive integers, and their sum

$$a+\beta+\gamma+\&c.=m.$$

COR. If the given multinomial be $(a+bx+cx^2+\&c.)^m$, then the general term is

$$\frac{\lfloor m}{\lfloor a \times \lfloor \beta \times \lfloor \gamma + \&c.} a^a (bx)^\beta (cx^2)^\gamma \ldots = \frac{\lfloor m}{\lfloor a \times \lfloor \beta \times \lfloor \gamma \times \&c.} a^a b^\beta c^\gamma \ldots x^{\beta+2\gamma+\&c.}$$

where the Indices are all positive Integers, and $a+\beta+\gamma+\&c.=m$.

In order then to find the whole expansion of such a multinomial, we should have to give a, β, γ, &c. all possible positive integral values, subject to the condition $a+\beta+\gamma+\&c.=m$. But it is often required to find only the coeff. of a certain given power of x in such an expansion, as, for instance, of x^n: and then we have only to take those values of a, β, γ, &c. which satisfy the two conditions $a+\beta+\gamma+\&c.=m$, $\beta+2\gamma+\&c.=n$.

Ex. 1. Find the coeff. of x^5 in the expansion of $(a-bx-cx^2)^6$.

Here we must take a, β, γ, so as to satisfy the two equations $\left.\begin{matrix} a+\beta+\gamma=6 \\ \beta+2\gamma=5 \end{matrix}\right\}$. It will be best to set down first the highest value which can be given to γ in the *second* equation, with the corresponding values of β and a, that of a being obtained from the *first* equation; then the next lower value of γ, and so on, as follows:

$\left.\begin{matrix} \gamma=2, & \beta=1, & a=3, \\ \gamma=1, & \beta=3, & a=2, \\ \beta=5, & a=1, & \end{matrix}\right\}$ hence the coeff. required is (setting outside the common factor of each of its terms, viz. $\lfloor 6 = 1.2.3.4.5.6$)

$$\lfloor 6 \left\{\frac{a^3(-b)(-c)^2}{\lfloor 3 . \lfloor 1 . \lfloor 2} + \frac{a^2(-b)^3(-c)}{\lfloor 2 . \lfloor 3 . \lfloor 1} + \frac{a(-b)^5}{\lfloor 1 \lfloor 5}\right\} = -60a^3bc^2 + 60a^2b^3c - 6ab^5.$$

100. But (ii) if m be *not* a positive integer, then we cannot modify the form of the first result, though all the rest of the reasoning will remain the same; and the general term will therefore be $\dfrac{m(m-1)\dots(\alpha+1)}{\lfloor\beta\times\lfloor\gamma\times\&c.}\,a^{\alpha}b^{\beta}c^{\gamma}\dots$, where $\alpha+\beta+\gamma+\&c.=m$, as before, α being negative or fractional, but β, γ, &c. all positive integers.

Ex. 1. Find the coeff. of x^4 in the expansion of $(a+bx-cx^2-dx^3)^{-\frac{1}{2}}$.

Here $\left.\begin{aligned}\alpha+\beta+\gamma+\delta&=-\tfrac{1}{2}\\ \beta+2\gamma+3\delta&=4\end{aligned}\right\}$ Beginning, as before, with setting down the highest value which can be given to δ in the *second* equation, and determining α always from the *first*, we have

$\left.\begin{array}{l}\delta=1,\ \beta=1,\ \alpha=-\frac{5}{2},\\ \gamma=2,\ \alpha=-\frac{5}{2},\\ \gamma=1,\ \beta=2,\ \alpha=-\frac{7}{2},\\ \beta=4,\ \alpha=-\frac{9}{2},\end{array}\right\}.$ Hence the coeff. required is (the factors in the num^r of each term beginning with the index $-\frac{1}{2}$, and gradually descending by unity, till they end with the value of $\alpha+1$, corresponding to the term)

$$\frac{-\frac{1}{2}(-\frac{3}{2})}{\lfloor 1 . \lfloor 1}\,a^{-\frac{5}{2}}b\,(-d)+\frac{-\frac{1}{2}(-\frac{3}{2})}{\lfloor 2}\,a^{-\frac{5}{2}}(-c)^2+\&c.$$

$$=-\tfrac{3}{4}a^{-\frac{5}{2}}bd+\tfrac{3}{8}a^{-\frac{5}{2}}c^2+\tfrac{15}{16}a^{-\frac{7}{2}}b^2c+\tfrac{35}{128}a^{-\frac{9}{2}}b^4.$$

Ex. 2. Find the coeff. of x^3 in $(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\&c.)^{-1}$.

Since $(x-\frac{1}{3}x^3+\frac{1}{5}x^5+\&c.)^{-1}=x^{-1}(1-\frac{1}{3}x^2+\frac{1}{5}x^4-\&c.)^{-1}$, this reduces itself to finding the coeff. of x^4 in $(1-\frac{1}{3}x^2+\frac{1}{5}x^4-\&c.)^{-1}$, or in $(1-\frac{1}{3}x^2+\frac{1}{5}x^4)^{-1}$, because the terms left out, (in which the index of x is higher than 4,) cannot possibly affect the coeff. of x^4 in the expansion. We have here then the equations

$$\begin{aligned}\alpha+\beta+\gamma&=-1, &&\text{whence } \gamma=1,\ \alpha=-2,\\ 2\beta+4\gamma&=4, && \qquad\ \ \beta=2,\ \alpha=-3;\end{aligned}$$

and the coeff. required is

$$\frac{-1}{\lfloor 1}(1)^{-2}.(\tfrac{1}{5})+\frac{(-1)(-2)}{\lfloor 2}(1)^{-3}.(-\tfrac{1}{3})^2=-\tfrac{1}{5}+\tfrac{1}{9}=-\tfrac{4}{45}.$$

Ex. 23.

Apply the Multinomial Theorem to obtain the first five terms of

1. $(a+bx+cx^2)^5$. 2. $(a-\frac{1}{2}bx+\frac{1}{3}cx^2)^7$. 3. $(1+2x-3x^2+x^4)^4$.

4. $(1+2x+3x^2+\&c.)^{-2}$. 5. $(1+x+x^2+\&c.)^{-\frac{1}{2}}$. 6. $(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\&c.)^{-3}$.

7. $(1-2x-x^2-\frac{4}{3}x^3-\frac{7}{3}x^4)^{-\frac{3}{2}}$. 8. $(a-bx-cx^2)^{-\frac{2}{3}}$. 9. $(1-2x+3x^2-\&c.)^{-\frac{3}{2}}$.

10. Obtain the fourth and fifth terms of $(a-bx-cx^2)^6$.

11. Obtain the third and fourth terms of $(a+bx-cx^2-dx^3)^{-\frac{1}{2}}$.

12. Find the coeff^s. of x^5 and x^7 in $(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\&c.)^{-1}$.

CHAPTER VII.

LOGARITHMS, AND EXPONENTIAL THEOREM.

101. Def. The *logarithm* of a number to a given base is *the index of that power to which the base must be raised, in order to become equal to the number:* so that, if a^x = any number N, then x is called *the logarithm of N to the base a*, which is denoted thus, $x = \log_a N$, or (if there be no occasion to mention the base) $x = \log N$.

Thus, since $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, &c., $\therefore \log_{10} 1 = 0$, $\log_{10} 10 = 1$, $\log_{10} 100 = 2$, &c.; and so also, since $a^0 = 1$, $a^1 = a$, we have $\log_a 1 = 0$, $\log_a a = 1$, whatever a may be.

102. By taking any *positive* number (except unity) for base we may express any *positive* number as some power of it.

Thus take $a = 10$, as above, and let $N = 2$; then, since $10^0 = 1$ and $10^1 = 10$, there is some value of x, if we could find it, between 0 and 1, such that $10^x = 2$: and, in point of fact, it may be shewn that this value is (to five places of decimals) .30103, so that $\log_{10} 2 = .30103$. It would of course be possible, though tedious, to verify this statement by expanding $10^{.30103}$ or $(1+9)^{\frac{30103}{100000}}$ to a sufficient number of terms by the Bin. Theorem: but we shall see below that such would be the case.

103. If we give N the successive values 1, 2, 3, &c., and register the corresponding values of x (which may be found by methods to be given hereafter), the table thus formed is called a 'Table of Logarithms to the base 10'.

The integral part of any logarithm is called the *Characteristic*, the decimal part is called the *mantissa*, or *handful*, as it were, thrown in over and above the characteristic. Of course when there is no integral part, the characteristic is 0.

104. If the base a be >1, then since $a^{-\infty} = 0$, $a^0 = 1$, $a^{\infty} = \infty$, we have $\log_a 0 = -\infty$, $\log_a 1 = 0$, $\log_a \infty = \infty$; and thus the log. of any number will lie between 0 and $+\infty$ or 0 and $-\infty$, that is, will be + or −, according as the number itself is > or < 1. Of course, the contrary will be the case if we suppose a to be <1.

105. Although, in reasoning generally about Logarithms, we may consider the base to be any positive number other than unity, yet in actual practice we shall have only to deal with (i) Logs. to the base 10, which are called *Common* Logs., and (ii) Logs. to the base e, where e denotes a certain number 2.7182818, (of which more hereafter,) which are called *Napierian* Logs., from the name of Lord Napier of Merchiston, in Scotland, who first invented Logarithms.

106. It will be seen below that series may be found, by means of which logarithms to the base e may be readily calculated. Among these will be $\log_e 10 = 2.3025850928$; and we shall now shew that, if the log. of any n°. to base e be multiplied by the factor $1 \div \log_e 10 = 1 \div 2.302\&c. = .4342944$, it will become the corresponding logarithm to the base 10.

For let x, y, be the logs. of any number N to any two bases a and b, that is, let $x = \log_a N$, $y = \log_b N$, and $\therefore a^x = N = b^y$: then, since $b = a^{\log_a b}$, we have $a^x = (a^{\log_a b})^y = a^{\log_a b \cdot y}$, and $\therefore$ $x = \log_a b \cdot y$, or $\log_a N = \log_a b \cdot \log_b N$. Hence, of course, it follows that $\log_e N = \log_e 10 \cdot \log_{10} N$, or $\log_{10} N = \dfrac{1}{\log_e 10} \times \log_e N$.

This constant multiplier, by which the two systems of logs. are connected, is called the *Modulus* of the system to base 10, and will be denoted in future by M.

COR. $\log_a N = \log_a b \cdot \log_b N = \log_a b \cdot \log_b c \cdot \log_c N = \&c.$, and so on, through any number of bases.

107. We will suppose then that, by means of the Nap. logs., we have formed a Table of Common Logarithms, and from these we will extract for our present use the following.

Logarithms, to base 10, *of all Prime Numbers from* 1 *to* 100.

No.	Logarithms.	No.	Logarithms.	No.	Logarithms.
2	0.3010300	29	1.4623980	61	1.7853298
3	0.4771213	31	1.4913617	67	1.8260748
7	0.8450980	37	1.5682017	71	1.8512583
11	1.0413927	41	1.6127839	73	1.8633229
13	1.1139434	43	1.6334685	79	1.8976271
17	1.2304489	47	1.6720979	83	1.9190781
19	1.2787536	53	1.7242759	89	1.9493900
23	1.3617278	59	1.7708520	97	1.9867717

108. Now the following are the properties of logs. which make them of singular value in diminishing the labour of Arithmetical calculations.

(i) Log $(mn) = \log m + \log n$, or the log. of any product = the sum of the logs. of the factors.

For if $m = a^x$, and $n = a^y$, then $mn = a^{x+y}$, and $\therefore \log(mn) = x + y = \log m + \log n$.

Hence also $\log(mnp) = \log(mn) + \log p = \log m + \log n + \log p$, &c.

(ii) Log $\frac{m}{n} = \log m - \log n$, or the log. of any quotient = the rem^r obtained by subtracting the log. of the divisor from that of the dividend.

For $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$, and $\therefore \log\left(\frac{m}{n}\right) = x - y = \log m - \log n$.

Hence $\log\frac{1}{m} = \log 1 - \log m = -\log m$, since $\log 1 = 0$.

(iii) Log $m^n = n \log m$, or the log. of any power of a number is obtained by multiplying the log. of the number by the index of the power.

For $m^n = (a^x)^n = a^{nx}$, and $\therefore \log(m^n) = nx = n \log m$.

By means of the above results, all Arithmetical operations of Multiplication, Division, Involution, and Evolution, may be converted into *Addition* and *Subtraction* of Logarithms.

Ex. 1. $\log 6 = \log 2 + \log 3 = .7781513$;
$\log 5 = \log 10 - \log 2 = 1 - \log 2 = .6989700$.

Ex. 2. $\log 100 = \log 10^2 = 2 \log 10 = 2$, $\log 1000 = 3$, &c.

Ex. 3. $\log 4 = \log 2^2 = 2 \log 2 = .6020600$;
$\log 18 = \log 2 + \log 9 = \log 2 + 2 \log 3 = 1.2552725$.

Ex. 4. $\log .07 = \log \frac{7}{100} = \log 7 - \log 100 = .8450980 - 2$, which is written thus $\bar{2}.8450980$, it being understood that in this position of the negative sign, it belongs only to the characteristic 2, and not to the mantissa, which is still positive.

Ex. 5. $\log 2.4 = \log \frac{24}{10} = \log 3 + \log 8 - 1 = .4771213 + .9030900 - 1 = .3802113$;

$\log .0023 = \log \frac{23}{10000} = 1.3617278 - 4 = .3617278 - 3 = \bar{3}.3617278$.

Ex. 6. $\log \frac{3}{97} = .4771213 - 1.9867717 = -1.5096504$, which may be written thus, $-2 + (1 - .5096504) = \bar{2}.4903496$.

109. It will be seen from [108 Ex. 6] how to convert a log. which is *wholly* negative into one that shall be negative only in its characteristic, viz. *by increasing the given negative characteristic by unity, and subtracting from unity the given mantissa.*

Thus, generally, if $-(C+m)$ represent a log. in which the characteristic (C) and mantissa (m) are both negative, then

$$-(C+m)=-C-m=-(1+C)+(1-m),$$

where the mantissa is now positive.

The decimal thus obtained by subtracting another from unity is called its *Arithmetical Complement*, and is most readily written down in practice by subtracting its last figure from 10 and the others from 9. A log. thus modified we may call a *Complementary* Logarithm, and denote by colog.

By the use of Comp. Logs. the Subtr^{n}. of a log. may be turned into Add^{n}: thus $\log \frac{3}{97} = \log 3 + \operatorname{colog} 97 = .4771213$

$$\begin{array}{r} .4771213 \\ +\,\bar{2}.0132283 \\ \hline =\bar{2}.4903496, \end{array}$$ as before.

But it is necessary to notice some peculiarities which occur in the use of such logs., whose characteristics only are negative, and of which instances are given in the next examples.

Ex. 1. $\log \frac{2}{17} + \log \frac{3}{19} = \log 2 + \operatorname{colog} 17 + \log 3 + \operatorname{colog} 19$

$$\begin{array}{r} =0.3010300 \\ +\,\bar{2}.7695511 \\ +\,0.4771213 \\ +\,\bar{2}.7212464 \\ \hline \bar{2}.2689488 \end{array}$$

Here there was an integer $+2$, arising from the sum of the positive mantissæ, which, combined with the sum of the characteristics -4, leaves -2 or $\bar{2}$.

Ex. 2. $\log \left(\frac{1}{13}\right)^3 = 3 \log \frac{1}{13} = 3 \operatorname{colog} 13 = \bar{2}.8860566$

$$\begin{array}{r} \bar{2}.8860566 \\ 3 \\ \hline \bar{4}.6581698 \end{array}$$

Here there was an integer $+2$, arising from the product of the positive mantissa by 3, which, combined with $3 \times \bar{2} = -6$, leaves -4 or $\bar{4}$.

Ex. 3. $\log \sqrt[7]{\frac{1}{71}} = \frac{1}{7} \operatorname{colog} 71 = \frac{1}{7}(\bar{2}.1487417) = \bar{1}.7355345.$

Here it was necessary to imagine the log. thrown into the equivalent form $\frac{1}{7}(\bar{7}+5.1487417)$, so that the negative characteristic may become a multiple of 7.

Ex. 4. Find the log. of $\{\sqrt{\tfrac{1}{2}} \times \sqrt[3]{\tfrac{2}{3}} \div \sqrt[4]{\tfrac{3}{4}}\}^{-\frac{3}{5}}$.

Here $\log \tfrac{1}{2} = \text{colog}\ 2 = \bar{1}.6989700$; $\qquad \therefore \tfrac{1}{2} \log \tfrac{1}{2} = \bar{1}.8494850$

$\log \tfrac{2}{3} = .3010300 + \bar{1}.5228787 = \bar{1}.8239087$; $\therefore \tfrac{1}{3} \log \tfrac{2}{3} = \bar{1}.9413029$

$-\tfrac{1}{4} \log \tfrac{3}{4} = +\tfrac{1}{4} \log \tfrac{4}{3} = \tfrac{1}{4}(.6020600 + \bar{1}.5228787) \qquad = 0.0312346$

$\bar{1}.8220225$

and, lastly, $-\tfrac{3}{5} \times \bar{1}.8220225 = \tfrac{1}{5}(3 - 2.4660675) = \tfrac{1}{5}(.5339325) = .1067865$

Ex. 24.

Obtain the logarithms of

1. 8, 9, 12, 20, 25, 60.
2. $\tfrac{1}{3}$, $\tfrac{1}{4}$, $\tfrac{2}{5}$, .03, $\tfrac{1}{30}$, .0033.
3. 1.8, 140, 1.44, .0625, $\tfrac{1}{1210}$.
4. 1.05, 10.6, $4\tfrac{1}{5}$, $4\tfrac{2}{5}$, .0111.
5. $\sqrt{1\tfrac{1}{2}}$, $\sqrt[3]{1\tfrac{1}{3}}$, $\sqrt{.1}$, $\sqrt[3]{.02}$, $(1.2)^{\frac{2}{3}}$.
6. $\dfrac{\sqrt{.122}}{\sqrt[3]{.123}}$.
7. $\dfrac{(1.02)^{-2}}{(2.01)^{-3}}$.
8. $(\tfrac{1}{7})^3$, $\tfrac{3}{4}\sqrt{1.1}$, $(.069)^{\frac{2}{5}}$, $(3\tfrac{1}{5})^{-\frac{1}{3}}$.
9. $\sqrt[4]{\tfrac{1}{2}} \times \sqrt[3]{\tfrac{2}{3}} \times \sqrt{\tfrac{3}{4}}$.
10. $\sqrt{\{\tfrac{1}{2}\sqrt[3]{(\tfrac{2}{3}\sqrt[4]{\tfrac{3}{4}})}\}}$.

There are, however, two observations to be made, which greatly facilitate the finding of Common logarithms.

110. (i) In the Common system having given $\log N$, we can find immediately $\log (N \times 10^n)$ or $\log (N \div 10^n)$, n being an integer, that is, we can find the log. of any number, *which differs from N only in the position of the decimal point.*

For $\log (N \times 10^{\pm n}) = \log N \pm n \log 10 = \log N \pm n$, and $\therefore$, since n is an integer, will differ from $\log N$ only in the characteristic, which will be increased or diminished by n, while both logs. will have the same mantissa.

Thus, suppose that we have given $\qquad \log 1362 = 3.1341771$:

then $\log 136200 = \log (1362 \times 10^2) = \log 1362 + 2 = 5.1341771$,

$\log 1.362 = \log (1362 \div 10^3) = \log 1362 - 3 = \ .1341771$,

$\log .001362 = \log (1362 \div 10^6) = \log 1362 - 6 = \bar{3}.1341771$.

111. (ii) In the Common system, the characteristic of the log. of any number may be written down at once by inspection.

For if the No. lie between 1 and 10, that is, between 10^0 and 10^1, its log. must lie between 0 and 1, and $\therefore$ its characteristic will be 0; so, if it lie between 10 and 100, that is, between 10^1 and 10^2, its characteristic will be 1; if it lie between 100 and 1000, that is, between 10^2 and 10^3, its characteristic will be 2; and generally, if it have n digits, that is, if it lie between 10^{n-1} and 10^n, its log. will lie between $n-1$ and n, and its characteristic will be $n-1$.

Again, if the No. lie between 1 and .1, that is, between 1 and $\frac{1}{10} = 10^{-1}$. its log. must lie between 0 and -1, and ∴ (with positive mantissa) its characteristic will be $\bar{1}$; so, if it lie between .1 and .01, that is, between 10^{-1} and 10^{-2}, the characteristic will be $\bar{2}$; if it lie between .01 and .001, that is, between 10^{-2} and 10^{-3}, the characteristic will be $\bar{3}$; and generally, if it have $n-1$ cyphers after the point, the characteristic will be $\bar{n}$.

It will be seen that both cases are comprised in the following Rule: Reckon the distance of the first *significant figure* from the *units-place:* if it be n places to the left, the characteristic will be $+n$; if n places to the right, $\bar{n}$.

112. Hence it appears that, in the Tables of Common Logs., it is only necessary to register the *mantissæ*, corresponding to certain *sequences* of figures, which look like numbers but are not really so, because the place of the decimal point is not fixed in them.

Thus in the Table given below, we have, opposite to the value 3570 for N, the mantissa 5526682, where 3570 is no number, but merely a sequence of figures, and the Table shews that for all such sequences, wherever we insert in them the decimal point, the mantissa is still 5526682: thus, determining the characteristic in each case by [111], we have log 3.570 or log 3.57 = .5526682, log 35700 = 4.5526682, log .00357 = $\bar{3}$.5526682.

113. The mantissæ of logarithms are registered in the best Tables to 7 decimal places, corresponding to sequences of 5 figures, as in the lines below, extracted from Hutton's Tables.

N.	0	1	2	3	4	5	6	7	8	9	D.	Pro.
3570	5526682	6804	6925	7047	7169	7290	7412	7534	7655	7777		122 1 12 2 24 3 37 4 49 5 61 6 73 7 85 8 98 9 110
71	7899	8020	8142	8263	8385	8507	8628	8750	8871	8993	122	
72	9115	9236	9358	9479	9601	9722	9844	9965	0087	0209		
73	5530330	0452	0573	0695	0816	0938	1059	1181	1302	1424		

Thus, to find the mantissa for the sequence $N = 35725$, we look horizontally along the third column for 3572, and vertically down under the figure 5, and thus find it to be 5529722, including the figures 552, which, being once printed (as in the first line) at the beginning of the line in which they first occur,

are understood to be repeated before each mantissa, until (as in the fourth line) they are replaced by 553. This change, however, actually begins with the last two mantissæ of the third line, where the dotted figures are introduced to mark this: thus, the mantissa for 35728 is 5530087.

In some books, instead of a dotted figure, a figure with a line above it, as $\bar{0}$, or a *smaller* figure, is used to mark the point where this change begins, if it happen not to be at the beginning of a line.

114. Although the mantissæ are only given in the Tables for sequences of five figures, yet they may be readily found for sequences of six or seven figures by the following considerations.

It will be shewn hereafter, that when the difference of two numbers is small compared with either of them, the diff. of their logs. is very nearly proportional to the diff. of the numbers.

Now let m, $m+D$, be the mantissæ for two consecutive numbers, N and $N+1$, each of five figures, $m+\delta$ the mantissa for a number $N+d$, lying between them, that is, having the same integral part as N, but one or more figures after the decimal point. Then, since the three numbers have all the same number of integral digits, they will have all the same characteristic, and so the diff. of their mantissæ will be the same as the diff. of their logs.: hence, by the above statement,

$$D : \delta :: 1 : d, \text{ or } \delta = Dd,$$

by means of which result we may find δ when d is given, or conversely, d when δ is given.

Ex. 1. To find log 35.7235.

Here $N = 35723$, $N+1 = 35724$, $N+d = 35723.5$, and $\therefore d = \frac{5}{10}$; and $D =$ the Difference of the mantissæ of 35723 and $35724 = 122$: hence for 35723.5, $\delta = \frac{5}{10}$ of $122 = 5 \times 12.2 = 61$; and thus, the whole mantissa for 35723.5 being $5529479 + 61 = 5529540$, we have $\log 35.7235 = 1.5529540$.

So for 35723.57 the diff. (δ) is $\frac{57}{100}$ of $122 = 57 \times 1.22 = 69.54 = 70$ nearly, and $\therefore \log .003572357 = \bar{3}.5529549$.

Ex. 2. To find the number corresponding to the log $\bar{2}.5528797$.

Here the next lower mantissa is that for 35717, viz. 5528750, and $\therefore \delta = 47$: hence, since the Diff. (D) between the mantissæ for 35717 and 35718 is 121, we have $d = \delta \div D = \frac{47}{121} = .38$, (it being

useless to go beyond two decimal places, for a reason that will appear hereafter;) and so if the *number* of five figures, referred to as N in the proof, be 35717, the *number* $N+d$ will be 35717.38; from which we see that the given mantissa corresponds [112] to the *sequence* 3571738, and ∴ the given logarithm (the characteristic being $\bar{2}$) corresponds to the *number* .03571738.

115. But the columns headed D. and Pro. (Proportional Parts) are intended to facilitate the calculation of such logs., and the converse operation of finding the corresponding number from the given log. The column D shews that the *prevailing* Diff. between two consecutive mantissæ in this neighbourhood is 122, as will be seen at once to be the case by looking at them, the Diff. being sometimes 121, but generally 122: further back in the Tables, it would have been 123, 124, &c., and further on, 121, 120, &c.

Now for each value of D there is formed what is called a Table of Proportional Parts, by multiplying $\frac{D}{10}$, that is, in this case 12.2, by 1, 2, 3, &c. successively, which give $12.2 = 12$ (nearly), $24.4 = 24$, $36.6 = 37$, &c., (as in the extract on page 68), the integer only being retained in each product, but increased by unity when the rejected decimal part is not less than $\frac{1}{2}$ or .5. The use of this will now be apparent: for since $\delta = Dd$, let α, β, &c. be the figures in order of the decimal d; then

$$\delta = D\left(\frac{\alpha}{10} + \frac{\beta}{100} + \&c.\right) = \alpha\frac{D}{10} + \beta\frac{D}{100} + \&c.$$

Now, in the Table of Pro. Parts, the first column contains the successive values 1, 2, 3, &c. which α or β might have, and the second contains the corresponding values of $\alpha\frac{D}{10}$. In order, therefore, to find $\alpha\frac{D}{10}$, we have only to glance at the Table and take down the number opposite to the value of α; and in order to find $\beta\frac{D}{100}$, we have only to take down $\frac{1}{10}$th of the number, opposite to the value of β.

Ex. 1. To find log 357.2357.

Here we have mantissa for	35723	=	5529479
diff. for	5	=	61
diff. for	7 = 8.5	=	9
			5529549

and ∴ the log. required is 2.5529549.

Ex. 2. To find the number corresponding to the log $\bar{1}.5529789$.

Here the mantissa next below the given one being that for 35725, viz. 5529722, we have $\delta = 67$, from which taking 61, which we see, by the Table of Pro. Parts, gives the *sixth* figure 5, we have still remaining 6, which, being $\frac{1}{10}$ of 60, shews that the *seventh* figure is 4 (nearly 5); hence the *sequence* required is 3572554, and the *number* (the characteristic being $\bar{1}$) is .3572554.

Ex. 25.

In these Ex. the Tables in pp. 64, 68, *are to be consulted.*

1. Find the logs. of 35.70925, .3572739, .003571246, 3.572804.
2. Find the Nos. whose logs. are
   3.5528742, $\bar{1}.5530895$, $\bar{4}.5527777$, .5530199.
3. Given log 2000.1=3.3010517, construct a Table of Pro. Parts; and find log 20.00094, and the No. whose log. is $\bar{2}.3010489$.
4. Given $\log 31.001 = 1.4913757$,
   find log 3100023, $31000\frac{1}{4}$, and $31\frac{3}{4000}$.
5. Find the value of $\sqrt{2\frac{1}{3}} \times \sqrt[3]{3\frac{1}{4}} \times \sqrt[4]{4\frac{1}{5}} \times \sqrt[5]{5\frac{1}{6}}$,
   given $\log 4.4985 = .6530677$, $\log 4498.6 = 3.6530774$.
6. Find the value of $\sqrt{357.1328} \div \sqrt[3]{35712.75}$,
   given $\log 57.386 = 1.7588060$, $\log .057387 = \bar{2}.7588135$.

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116. We shall now explain the method of calculating logs. to the base e.

Exponential Theorem: To expand a^x in a series of ascending powers of x.

$$a^x=\{1+(a-1)\}^x=1+x(a-1)+\frac{x(x-1)}{1.2}(a-1)^2+\frac{x(x-1)(x-2)}{1.2.3}(a-1)^3+\&c.$$

$$=1+x\left\{(a-1)+\frac{x-1}{2}(a-1)^2+\frac{(x-1)(x-2)}{1.2.3}(a-1)^3+\&c.\right\}$$

$$=1+x\{A+Bx+Cx^2+\&c.\}\text{ suppose,}$$

where A, the sum of those terms within the bracket which are independent of x, is easily found by putting $x=0$ within the bracket, when there remains

$$(a-1)+\frac{-1}{2}(a-1)^2+\frac{(-1)(-2)}{1.2.3}(a-1)^3+\&c.,\text{ or }(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\&c.$$

which is therefore the value of A.

Hence, $a^x=1+Ax+Bx^2+Cx^3+\&c.$, where A, B, C, &c. are functions of a, altogether independent of x, and will therefore remain the same for the same value of a, however we change that of x.

Hence $a^{x+h} = 1 + A(x+h) + B(x+h)^2 + C(x+h)^3 + \&c.$:

but $a^{x+h} = a^x . a^h = \{1 + Ax + Bx^2 + Cx^3 + \&c.\} a^h$;

therefore, equating coeff^s of x in these identical values of a^{x+h}, we have $A + 2Bh + 3Ch^2 + \&c. = Aa^h = A\{1 + Ah + Bh^2 + Ch^3 + \&c.\}$ hence, equating coeff^s of different powers of h in the above, we have $2B = A^2$, $3C = AB$, &c., or $B = \dfrac{A^2}{1.2}$, $C = \dfrac{AB}{3} = \dfrac{A^3}{1.2.3}$, &c.;

so that we have
$$a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \&c.,$$
where $A = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.$

117. Since the above result is true for all values of x, take x such that $Ax = 1$; then $x = \dfrac{1}{A}$, and $a^{\frac{1}{A}} = 1 + 1 + \dfrac{1}{1.2} + \dfrac{1}{1.2.3} + \&c.$, $= 2.7182818$ &c., which number it is usual to denote by e; hence $a^{\frac{1}{A}} = e$, or $a = e^A$, and therefore $A = \log_e a$, and so we have
$$a^x = 1 + (\log_e a)\,x + (\log_e a)^2 \frac{x^2}{1.2} + (\log_e a)^3 \frac{x^3}{1.2.3} + \&c.:$$
whence also, writing e for a, we get, since (101) $\log_e e = 1$,
$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$$

118. We have
$$\left(1 + \frac{x}{n}\right)^n = 1 + n\frac{x}{n} + \frac{n(n-1)}{1.2} . \frac{x^2}{n^2} + \&c. = 1 + x + \frac{n-1}{n} . \frac{x^2}{1.2} + \&c.$$
$$= 1 + x + \left(1 - \frac{1}{n}\right)\frac{x^2}{2} + \left(1 - \frac{1}{n}\right)\left(\frac{1}{2} - \frac{1}{n}\right) \cdot \frac{x^3}{3} + \&c.$$

Now, as n increases, it is plain that $\dfrac{1}{n}$ decreases, and tends to zero as its *Limit:* hence we see that, as n increases, the Limit of $\left(1 + \dfrac{x}{n}\right)^n$ is $1 + x + \dfrac{x^2}{1.2} + \dfrac{x^3}{1.2.3} + \&c.$ or e^x.

119. By [116] we have $(e^x - 1)^n = (x + \frac{1}{2}x^2 + \&c.)^n = x^n + \&c.$ (i): but we have also $(e^x - 1)^n = e^{nx} - n\,e^{(n-1)x} + \frac{1}{2}n(n-1)\,e^{(n-2)x} - \&c.$ (ii), in which each of the quantities e^{nx}, $e^{(n-1)x}$, &c. may be expanded by [116], so that the whole coeff. of x^r in (ii) will be
$$\frac{n^r}{1.2\ldots r} - n\frac{(n-1)^r}{1.2\ldots r} + \frac{1}{2}n(n-1)\frac{(n-2)^r}{1.2\ldots r} - \&c.:$$

but in (i) the coeff. of x^r is zero, if $r < n$, and it is 1, if $r = n$; hence, since the two expressions for $(e^x - 1)^n$ must be identical,

$$n^r - n(n-1)^r + \tfrac{1}{2}n(n-1)(n-2)^r - \&c. = 0, \text{ if } r < n,$$

and $n^n - n(n-1)^n + \tfrac{1}{2}n(n-1)(n-2)^n - \&c. = 1.2.3 \ldots n.$

120. In the value of $\log_e a = A = (a-1) - \frac{1}{2}(a-1)^2 + \&c.$, write $1 + x$ for a, and $\therefore$ x for $a - 1$;

then $\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \&c. \ldots\ldots\ldots \text{(i)}.$

Hence we might proceed to find the logarithms of numbers:

$$\text{thus } \log_e 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \&c., = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \&c.,$$

but the series thus obtained are not sufficiently convergent, as it would take very many terms to obtain the logarithm accurately to 6 or 7 decimal places.

121. A more convergent series, however, may be obtained as follows from the above.

Since $\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \&c.$,

$\therefore$ $\log_e(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \&c.$, writing $-x$ for x;

and $\therefore$ $\log_e(1+x) - \log_e(1-x)$ or $\log_e \dfrac{1+x}{1-x} = 2\{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c.\}$.

In this series write $\dfrac{m-n}{m+n}$ for x, and $\therefore$ $\dfrac{m}{n}$ for $\dfrac{1+x}{1-x}$;

$$\therefore \log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \&c.\right\} \ldots\ldots (\alpha).$$

$$\text{If } n = 1, \text{ then } \log_e m = 2\left\{\frac{m-1}{m+1} + \frac{1}{3}\left(\frac{m-1}{m+1}\right)^3 + \frac{1}{5}\left(\frac{m-1}{m+1}\right)^5 + \&c.\right\} \ldots (\beta),$$

a series, from which the logs. of low primes may be found.

$$\text{Thus } \log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \&c.\right\} = .6931471806,$$

as may be seen by taking nine term of the series.

122. Again, if in (α) $m = n + 1$, we have

$$\log_e \frac{n+1}{n} = \log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \&c.\right\} \ldots (\gamma),$$

a very convergent series, by means of which, having given the logarithm of one of two consecutive numbers, n and $n+1$, we may find that of the other.

Thus $\log_e 9 - \log_e 8 = 2\log_e 3 - 3\log_e 2 = 2\left\{\frac{1}{17} + \frac{1}{3.17^3} + \&c.\right\}$;
and by means of *four* terms of this series, having given $\log_e 2$ above, we may find $\log_e 3 = 1.0986122884$.

And $\log_e 5 - \log_e 4 = \log_e 5 - 2\log_e 2 = 2\left\{\frac{1}{9} + \frac{1}{3.9^3} + \&c.\right\}$,
by taking *five* terms of which series we get $\log_e 5 = 1.6094379122$, and adding $\log_e 2$ to this, we have $\log_e 10 = 2.3025850928$, and thence, by common div^n, the modulus $M = 1 \div \log_e 10 = .4342944819$.

123. Once more in (γ) write x^2 for $n+1$, and $\therefore$ $2x^2-1$ for $2n+1$; then

$$\log_e \frac{x^2}{x^2-1} = 2\log_e x - \log_e(x+1) - \log_e(x-1)$$
$$= 2\left\{\frac{1}{2x^2-1} + \frac{1}{3(2x^2-1)^3} + \&c.\right\} \ldots\ldots (\delta),$$

a very rapidly converging series, by means of which, having given the logs. of any two of three consecutive numbers, $n+1$, n, $n-1$, we may find that of the other.

Thus, taking the numbers 5, 6, 7, of which the logarithms of 5 and 6 ($=2\times 3$) are now known, we have

$$2\log_e 6 - \log_e 7 - \log_e 5 = 2\left\{\frac{1}{71} + \frac{1}{3(71)^3} + \&c.\right\},$$

whence we may find $\log_e 7$, &c.

Or the same might be found by (γ) as follows:

$\log_e 50 - \log_e 49$, or $(\log_e 2 + 2\log_e 5) - 2\log_e 7 = 2\{\tfrac{1}{79} + \&c.\}$.

124. By means of the above (or other similar formulæ) a table of Napierian logs. may be formed; and then these may be converted into logs. to the base 10 or to any other base, by multiplying each by the proper modulus.

Thus $\log_{10} 2 = M\log_e 2 = .4342944819 \times .6931471806 = .3010300$; and so we might find, if desired, $\log_3 2 = (1 \div \log_e 3) \times \log_e 2$.

Or having once obtained $\log_e 10$ and, by means of it, the *modulus*, we may write at once in (a)

$$\log_{10}\frac{m}{n} = M\log_e\frac{m}{n} = 2M\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \&c.\right\};$$

and so we get, corresponding to the formulæ (β), (γ), (δ), above,

$$\log_{10} m = 2M\left\{\frac{m-1}{m+1} + \&c.\right\},\quad \log_{10}(n+1) - \log_{10} n = 2M\left\{\frac{1}{2n+1} + \&c.\right\},$$

and $2\log_{10} x - \log_{10}(x+1) - \log_{10}(x-1) = 2M\left\{\frac{1}{2x^2-1} + \&c.\right\}$,

and thus may calculate immediately the common logarithms, without finding the Napierian.

125. We are now able to prove the statement made in [114].

$$\text{For } \delta = \log_{10}(N+d) - \log_{10} N = \log_{10}\frac{N+d}{N} = M\log_e\left(1+\frac{d}{N}\right)$$

$$= \text{by (i) } M\left\{\frac{d}{N} - \&c.\right\} = \frac{M}{N}d \text{ nearly,}$$

when d is small compared with N: that is, $\delta \propto d$, or the increase of the log. is proportional to the increase of the number.

$$\text{Also} \quad D = \log_{10}\frac{N+1}{N} = M\log_e\left(1+\frac{1}{N}\right) = \frac{M}{N} \text{ nearly;}$$

now $M = .43\&c.$, and is $\therefore < \frac{1}{2}$, while N, if a number of five places, is > 10000; hence it follows that $D < \frac{1}{20000} < .00005$ in every case, and $\therefore \frac{D}{1000}$ would at the utmost only have its first significant figure in the *eighth* place of decimals, and so $\gamma\frac{D}{10000}$ (see [115]) could only, if at all, affect the *seventh* or last figure of the mantissa, but would not generally affect it at all: thus to 7 places of decimals the logs. of 35714.23 and 35714.232198&c. would probably coincide, the diff. appearing in the 8th and following places. Hence with Tables which give mantissæ only to 7 figures, we can only expect to find the numbers which correspond to given logs. correct in their first seven figures.

126. Equations of the following kind, in which the unknown quantity occurs in the form of an index, are called *Exponential* Equations, and usually require logarithms for their solution.

Ex. 1. $a^x = b$. Here $x \log a = \log b$, or $x = \frac{\log b}{\log a}$.

Ex. 2. $2^{3x} . 5^{2x-1} = 4^{5x} . 3^{x+1}$.

$$\text{Here } 3x\log 2 + (2x-1)\log 5 = 5x\log 4 + (x+1)\log 3,$$
$$= 10x\log 2 + (x+1)\log 3:$$
$$\text{or } x(2\log 5 - 7\log 2 - \log 3) = \log 3 + \log 5,$$
$$\therefore x = \frac{1.1760913}{\bar{2}.8136087} = -\frac{1.1760913}{1.1863913} = -.991 \text{ nearly.}$$

Ex. 26.

1. $ab^x = c$.
2. $a^{b^x} = c$.
3. $5^x = 800$.
4. $(\sqrt[m]{a^n})^x = b^{nx-a}$.
5. $a^x b^y = c,\ my = nx$.
6. $(a+b)^2 (a^2-b^2)^{x-1} = (a-b)^{2x}$.
7. $x^x - x^{-x} = 3(1 + x^{-x})$.
8. $(2\tfrac{1}{2})^{2x} \times 5^{3x-5} = (1\tfrac{1}{4})^{x+1} \times (1\tfrac{3}{5})^{2-x}$.
9. $x^y = y^x,\ x^a = y^b$.
10. $a^{4x} + a^{2x} = a^{6x}$.
11. $a^x + a^{-x} = b$.
12. $x^{\frac{1}{2}(\sqrt[4]{x}+\sqrt[4]{y})} = y^{\frac{4}{3}},\ y^{\frac{1}{4}(\sqrt[4]{x}+\sqrt[4]{y})} = x^{\frac{1}{3}}$.

CHAPTER VIII.

NOTATION, INTEREST, &c.

127. A No. in any scale may be multiplied by any power of the radix of that scale, by annexing as many cyphers as there are units in its index.

For if $$N = a_n r^n + a_{n-1} r^{n-1} + \&c. + a_0,$$
then $$N.r^p = a_n r^{n+p} + a_{n-1} r^{n+p-1} + \&c. + a_0 r^p,$$
where the coeff[s] of r^{p-1}, r^{p-2}, &c. being wanting, we shall have cyphers in the places of the corresponding digits.

This is, of course, the case in Arithmetic with any power of 10.

128. *The greatest No. that can be expressed with* n *digits in the scale of* r *is* $r^n - 1$, *the least is* r^{n-1}.

For N will be *greatest*, when p_{n-1}, p_{n-2}, &c. have each of them their greatest possible value, i.e. $r - 1$; so that
$$N = (r-1)(r^{n-1} + r^{n-2} + \&c. + r^2 + r + 1) = r^n - 1;$$
and *least* when $p_{n-1} = 1$, and p_{n-2}, p_{n-3}, &c. all vanish, so that $N = r^{n-1}$.

Ex. The greatest and least Nos. that can be expressed in the scale of 7 with 7 digits, are $7^7 - 1$ and 7^6, or 823542 and 117649, which, expressed in the scale of 7, are 6666666 and 1000000.

129. Since 10 is div. by 2, 10^2 or 100 by 4, 10^3 or 1000 by 8, &c., it will be seen that whenever the No. expressed by the last *one*, *two*, *three*, &c. digits of any common No. is div. by 2, 4, 8, &c. the No. itself will be so divisible. Thus 23456 may be written $23 \times 1000 + 456$, and is therefore div. by 8, since 456 is so divisible. (See *Arithmetic*, p. 20, Note.)

Generally, any common No. is div. by 2^p, if the No. expressed by the last p digits be so divisible.

130. It follows from [9] that if $N = p_{n-1} r^n + p_{n-2} r^{n-1} + \&c.$ be divided algebraically by $r - a$, the rem[r] thus obtained, or R, will be $p_{n-1} a^n + p_{n-2} a^{n-1} + \&c.$ Hence

(i) if $a = 1$, then $R = p_{n-1} + p_{n-2} + \&c.$ = the sum of the digits; and therefore (supposing $r = 10$) any common number and the sum of its digits, when divided by 9, will leave the same rem[r]:

(ii) if $a = -1$, then $\pm R = p_{n-1} - p_{n-2} + \&c.$; and, therefore, any common number and the diff. of its digits in odd and even places, when divided by 11, will leave the same rem^r.

Hence any common No. will be divisible exactly by 9 or 11, if the sum of its digits in the one case, or the diff. of its odd and even digits in the other, be so divisible. The former of these statements manifestly comprehends a similar statement for 3 as well as 9. (See *Arithmetic*, as before.)

131. Hence we may prove the common process of *casting out nines*, in order to test the truth of a sum in Mult^n. (*Arithmetic*, p. 6.)

For let P and Q be any two Nos.; and let $P = 9p + a$, $Q = 9q + \beta$, where a, β, the rem^{rs} upon dividing P and Q by 9, may be found [130] by *adding the figures* of P and Q, and dividing the results by 9. Set these down in the upper and lower angles of a cross, and then, dividing $a\beta$ by 9, set down the rem^r in a third angle of the cross.

Now $PQ = 81pq + 9aq + 9\beta p + a\beta$; and, therefore, the rem^r upon dividing PQ, the *product* of P and Q, or [130] the *sum of the figures* in PQ, by 9, will be the same as that left on dividing $a\beta$ by 9. Set this rem^r then in the fourth angle of the cross: it should be the same as that just before set down in the third.

This method, however, only shews when we are *wrong*, but not always that we are *right*, in our sum; for if we have omitted a 9, or any multiple of 9, or misplaced figures, &c., these errors would not be detected by it.

132. So too we may prove a sum in Division. (*Arithmetic*, p. 9.)

For let D be the dividend: P, Q, the divisor and quotient, R the rem^r; then $D = PQ + R$, and $D - R = PQ$: hence $D - R$ is the product of P and Q; therefore, by [131], cast out nines from P and Q, and let the rem^{rs} be a, β; then cast out nines from $a\beta$, and from $D - R$, and the rem^{rs} will be the same.

The method will apply also to cases of Involution and Evolution, observing in the latter to subtract the rem^r (if any) from the given number, as in the case of Div^n: thus if D be the given number, Q its square root with rem^r R, then $D - R = Q^2$; therefore, casting nines out of Q with rem^r a, and then out of aa or a^2, the rem^r from the latter will be the same as from $D - R$.

133. It is plain that the same tests may be similarly shewn to apply in any scale, upon casting out the corresponding value of $r-1$.

Thus in (189 Ex. 2, 3, 4) we have $\begin{array}{ccc} & 4 & \\ 2 & \times & 2 \\ & 8 & \end{array}$, $\begin{array}{ccc} & 2 & \\ 4 & \times & 4 \\ & 6 & \end{array}$, $\begin{array}{ccc} & 1 & \\ 0 & \times & 0 \\ & 0 & \end{array}$, $\begin{array}{ccc} & 4 & \\ 1 & \times & 1 \\ & 4 & \end{array}$;

where (i) for $68 \times 71 = 4378$ (*und.*) we cast out the *tens;*

(ii) for $82 \times 86 = 7823$ (*non.*) we cast out the *eights;*

(iii) subtracting first the rem^r,
for $234340 \div 414 = 310$ (*quin.*) we cast out the *fours;*

(iv) substituting first the rem^r,
for $122024 = 252 \times 252$ (*sen.*) we cast out the *fives.*

134. Hitherto we have spoken only of integral Nos.; but we may extend the process to *fractional* ones by introducing *negative* powers of the index, that is, we may express any No. whatever, by means of any given radix r, in the form

$$N = p_{n-1}r^n + \&c. + p_1 r + p_0 + q_1 r^{-1} + q_2 r^{-2} + \&c.$$

where q_1, q_2, &c. are used to express the digits to the right of a_0, each of which, like the others, will be $< r$. We need only consider the case of a *proper* fraction; for the integral part (if any) of a given No. may be expressed as before.

Let $\frac{a}{b}$ be such a fraction, and $\frac{a \times r}{b} = q_1 + \frac{r_1}{b}$, $\frac{r_1 \times r}{b} = q_2 + \frac{r_2}{b}$, &c.

q_1, q_2, &c. being the integral quotients, and r_1, r_2, &c. the remainders, obtained by dividing $a \times r$, $r_1 \times r$, &c., successively by b: then, since a, r_1, &c. are each $< b$, $\therefore q_1$, q_2, &c. are each $< r$, and

$$\frac{a}{b} = \frac{q_1}{r} + \frac{1}{r} \cdot \frac{r_1}{b} = \frac{q_1}{r} + \frac{1}{r}\left(\frac{q_2}{r} + \frac{1}{r} \cdot \frac{r_2}{b}\right) = \frac{q_1}{r} + \frac{q_2}{r^2} + \frac{1}{r^2} \cdot \frac{r_2}{b} = q_1 r^{-1} + q_2 r^{-2} + \&c.$$

Ex. Express $56\frac{7}{13}$ in the nonary scale.

Here $56 = 62$ (nonary); and for $\frac{7}{13}$ we have, as above,

$$\frac{7 \times 9}{13} = 4\tfrac{11}{13},\quad \frac{11 \times 9}{13} = 7\tfrac{8}{13},\quad \frac{8 \times 9}{13} = 5\tfrac{7}{13},\quad \frac{7 \times 9}{13} = 4\tfrac{11}{13}, \&c.,$$

after which the same quantities will evidently *recur* continually: hence

$$56\tfrac{7}{13} = 6 \times 9 + 2 + 4 \times 9^{-1} + 7 \times 9^{-2} + 5 \times 9^{-3} + 4 \times 9^{-4} + \&c.$$

or, as such a quantity is usually written, $= 62.475\dot{4}7\dot{5}$ &c. $= 62.\dot{4}7\dot{5}$, the dots being used both after the units-figure, and to express the circulation, just as in common decimals, which, it is plain, are only

a particular case of the above, when $r = 10$. And all such quantities may be dealt with in Addition, &c., exactly like common Decimals, only taking account of the value of the radix.

135. We may, in fact, operate upon $\frac{7}{13}$ just as we do to bring a Vulgar Fraction to a Decimal, if we express first its num^r and den^r in the nonary scale: thus $\frac{7}{13} = \frac{7}{14}$ in the nonary, (observe, not $\frac{1}{2}$, for 14 is not here *fourteen*,) with which we proceed as below.

$$\begin{array}{r} 14)\ 7.0\ (.475 \\ \underline{57} \\ 120 \\ \underline{111} \\ 80 \\ \underline{72} \\ 7 \end{array}$$

It will be seen that we here multiply the original num^r and each rem^r successively (just as in the case of a Vulgar Fraction) by 10 (that is, precisely in accordance with the foregoing rule, by the radix r, in this case *nine;*) and then the figures in the quotient, being the successive *integral-quotients*, are the digits in order, as before.

Ex. Express $\frac{79}{343}$ and .5 in the scale of 7.

Here $\frac{79}{343} = \frac{142}{1000}$ (septenary) = .142: and for .5 we may use either of the methods, which have been above given:

$$\begin{array}{r} .5 \\ \underline{7} \\ 3.5 \\ \underline{7} \\ 3.5 \end{array}$$

thus $.5 = \frac{1}{2}$ (denary) $= \frac{1}{2}$ (septenary) = .333&c.; or thus, which gives the same result, taking for digits the integral figures 3, 3, &c., obtained in this process, which, it is easily seen, consists in multiplying (according to the rule) the num^r of the original fraction $\frac{5}{10}$ and each rem^r by 7, and dividing by the den^r 10, the division being indicated by the decimal point.

136. It will be seen that the results of (190-193), in which, from their being so commonly in use, we have spoken expressly of *decimal* fractions, may be applied to fractions in any scale whatever by considering 10 to stand for $1 \times r + 0$, that is for *any* radix r.

The *statement*, however, of (193) must be modified for other scales, though the *proof* holds good for all: thus in the scale of 4, $10^q - 1$ will be expressed by q *threes*, in that of 8 by q *sevens*, &c.

137. The last result might have been proved as follows:

$$N = \frac{P}{10^p} + \frac{Q}{10^{p+q}} + \frac{Q}{10^{p+2q}} + \&c. = \frac{P}{10^p} + \frac{Q}{10^{p+q}}\left\{1 + \frac{1}{10^q} + \frac{1}{10^{2q}} + \&c.\right\}$$

$$= \frac{P}{10^p} + \frac{Q}{10^{p+q}} \cdot \frac{10^q}{10^q - 1}\,(146) = \frac{P}{10^p} + \frac{Q}{10^p(10^q - 1)} = \frac{(10^q.P + Q) - P}{10^p(10^q - 1)},$$

where it will be observed that, in the num^r, $10^q.P + Q$ is the same quantity as was expressed in (193) by PQ.

Ex. $2.5 \times .03$ (senary) $= .123$, $.07\dot{2}\dot{5}$ (nonary) $= \frac{717}{8800} = \frac{583}{6480}$ (denary).

138. *To shew that in the case of an irreducible fraction, the figures of the equivalent decimal must recur, and the number of figures in the period must be less than the denominator.*

For [134] if $\frac{a \times 10}{b} = q_1 + \frac{r_1}{b}$, $\frac{r_1 \times 10}{b} = q_2 + \frac{r_2}{b}$, &c., then the integral quotients q_1, q_2, &c. will be the successive figures of the decimal. Now here, whenever any one of the rem^{rs} is repeated, it is plain that the following quotient and rem^r will also be repeated, and therefore the following again, and so on; that is, the *period* will begin: and, therefore, as the rem^{rs}, (the divisor being b,) can only be 1, 2, 3, &c. $(b-1)$, it follows that there *might* be $b-1$ different rem^{rs}, *i.e.* there might be $b-1$ figures in the period, but no more.

139. If b be *prime* to 10, the period will begin immediately after the decimal point.

For suppose the same rem^r r_m to come over again, so that $r_n = r_m$; then $10r_{m-1} = bq_m + r_m$, $10r_{n-1} = bq_n + r_n$; $\therefore$ since $r_m = r_n$, we have $10(r_{m-1} \sim r_{n-1}) = b(q_m \sim q_n)$, which equation (since b is prime to 10) can only be satisfied by the quantities, $r_{m-1} \sim r_{n-1}$, $q_m \sim q_n$, being either each zero, or equimultiples of b and 10. But each of the rem^{rs} must be < the div^r b, and each of the quotients [138] must be < 10; hence $r_{m-1} \sim r_{n-1}$ must be $< b$, and $q_m \sim q_n$ must be < 10, and $\therefore$ we must have each of these differences = zero, that is, $r_{m-1} = r_{n-1}$, $q_m = q_n$.

Hence, in like manner, we shall have also $r_{m-2} = r_{n-2}$, $r_{m-3} = r_{n-3}$, &c., till we come back to $a = r_{n-m}$; so that the original num^r is repeated, and the period begins with the first digit after the point.

140. If b be of the form $2^m 5^n c$, then the period will begin after the m^{th} or n^{th} decimal place, according as m or n is greatest, and will consist of not more than $c-1$ figures.

For (supposing $m > n$) we have $\frac{a}{b} = \frac{a.5^{m-n}}{10^m.c}$: now $\frac{a.5^{m-n}}{c}$ may be converted into a decimal, in which (since c is prime to a, and also to 10, and therefore to 5^{m-n}) the period will begin immediately after the point, and will consist of not more than $c-1$ digits; and in $\frac{5^{m-n}.a}{10^m.c}$, the effect of the additional factor in the den^r will be only to carry the decimal point m places to the right. Similarly, if $n > m$.

N.B. This assumes that if $\frac{a.5^{m-n}}{c}$ should be a *mixed* No., the num^r of the proper fraction it contains will still be prime to c, since it is only to this fraction that [139] refers. This is easily seen; for let any fraction $\frac{a}{b} = q + \frac{r}{b}$, and suppose $\frac{r}{b}$ reducible, so that r and b have a common measure, d; then (63) d will also divide $qb + r$ or a, and therefore a could not have been *prime* to b.

141. It is plain from (194) that in the scales of 6, 7, 8, &c. all fractions can be expressed with *terminating* digits, whose den^{rs} are of the form $2^m 3^n$, 7^m, 2^m, respectively. This shews the advantage of taking a *composite* No. as radix, since it allows of more fractions being expressed with terminating digits. Moreover, it is plain that *within given limits* more Nos. will be found of the form $2^m 3^n$, than of the form $2^m 5^n$: hence 6 would have been preferable to 10 in this respect, but would be inconveniently small as a radix for expressing large Nos. The radix 12, however, which is composed of the same small factors, 2 and 3, is not liable to the same objection; and is perhaps the best that could have been chosen. Almost all nations however, probably from reference to the fingers, have chosen to reckon by a *decimal* notation.

142. Since 1 ft. = 12 in., this suggests the employment of the *duodecimal* scale, in the calculation of lengths, areas, and solids.

Ex. Find the area of a floor, 30 ft. 3 in. long by 18 ft. 7 in. wide.

$$\begin{array}{ccc} & 0 & \\ 0 & \times & 0 \\ & 3 & \end{array} \qquad \begin{array}{r} 26.3 \\ \underline{16.7} \\ 157\,9 \\ 1316 \\ 263 \\ \hline 3tt.19 \end{array}$$

Here the length and breadth, expressed duodecimally, are 26.3 ft. and 16.7 ft.; and to *prove* the result, we have cast out *elevens* by the side.

Since $3tt$ (duod.) = 562 (den.), and .19 or $\frac{19}{100}$ (duod.) = $\frac{21}{144}$ (den.), we may write the Answer (noticing that 1 sq. in. = $\frac{1}{144}$ sq. ft.) 562 sq. ft. 21 in.; or we may denote it thus $562.1'.9''$, where $1'$, $9''$, (read 1 *minute*, 9 *seconds*) denote $\frac{1}{12}$, $\frac{9}{144}$, respectively.

Ex. 27.

1. Express and multiply together $\frac{3}{8}$, $2\frac{1}{12}$, $3\frac{1}{5}$ in the senary scale.
2. Express $.04\dot{3}$ (quinary) and $.26\dot{3}\dot{1}$ (septenary) as fractions.
3. Express 23.32 in the quat., oct., and duodenary scales.
4. Express $\frac{1}{100}$ in the binary, ternary, and undenary scales.

5. Express $\frac{16}{27}$, $2\frac{17}{32}$, $\frac{24}{36}$, in the senary and duodenary scales: multiply them together in each scale, and transform each result to the other scale.

6. Reduce $.06\dot{5}\dot{4}$ to a vulgar fraction, supposing it to be a number given in each of the first four scales which employ the digit 6.

7. Find the area of a floor 20 ft. 10 in. long by 13 ft. 4 in. broad, and find the side and diagonal of a square of the same dimensions.

8. How many yards of matting, 2 ft. 3 in. wide, will be wanted for a square room, whose side is 18 ft. 9 in.?

9. What is the length of a room, whose breadth is 11 ft. 11 in., and which it takes 17 sq. yds. 2 ft. 131 in. of drugget to cover?

10. How many cubic ft. of water may be contained in a vessel with square base, whose side is 6 ft. and height 1 ft. 5 in.?

11. Find the solid content of a beam of timber 15 ft. long, 2 ft. 3 in. wide, and 1 ft. $2\frac{1}{2}$ in. thick.

12. What is the length of a room, whose width is 10 ft. 4 in. and height 10 ft. 6 in., and which contains 3038 cubic ft. of air?

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143. If two Nos., P and Q, have p and q digits respectively, their product PQ will have either $p+q$ or $p+q-1$.

For by [128] $PQ < r^p . r^q$ and $> r^{p-1} . r^{q-1}$, that is, $< r^{p+q}$ and $> r^{p+q-2}$; and therefore will have $p+q$ or $p+q-1$ digits.

So, if R have r digits, then $PQR < r^{p+q+r} > r^{p+q+r-3}$, and therefore will have $p+q+r$, or $p+q+r-1$, or $p+q+r-2$ digits.

And, generally, if there be m Nos., the sum of whose digits $p+q+r+$ &c. $= n$, then their continued product will have from n to $n-(m-1)$ digits.

144. On the same supposition, $P \div Q$ will have $p-q$ or $p-q+1$ digits.

For $P \div Q < r^p \div r^{q-1}$ and $> r^{p-1} \div r^q$, that is, $< r^{p-q+1}$ and $> r^{p-q-1}$, and therefore will have $p-q$ or $p-q+1$ digits.

N.B. The preceding reasonings will evidently hold, if any of p, q, &c. should be *negative*, in which case there would be no digits in the corresponding No. to the *left* of the point: thus if a number N begins with q_3, the third *right-hand* digit, then $N < r^{-2} > r^{-3}$, and the above reasoning would apply, by taking -2, for the No. of digits in N, that is, by considering it to have as many digits *negatively*, as it has cyphers after the point.

145. Hence if a No. N have n digits, N^2 will have $2n$ or $2n-1$, N^3 will have $3n$, $3n-1$, or $3n-2$, &c.; and, generally, N^m will have from mn to $mn-(m-1)$ digits.

Again, if $\sqrt{N}$ have x digits, then N will have $2x$ or $2x-1$; $\therefore n$ is either $2x$ or $2x-1$, and $x = \frac{1}{2}n$ or $\frac{1}{2}(n+1)$, whichever is integral.

So if $\sqrt[3]{N}$ have x digits, then n is either $3x$, or $3x-1$, or $3x-2$; and $x = \frac{1}{3}n$, or $\frac{1}{3}(n+1)$, or $\frac{1}{3}(n+2)$, whichever is integral.

And, generally, if $\sqrt[m]{N}$ have x digits, then $x = \dfrac{n}{m}$, or $\dfrac{n+1}{m}$, &c. or $\dfrac{n+(m-1)}{m}$, whichever is integral.

It will be observed that the above values for x, in the case of the square and cube root, correspond, as they should, to the No. of dots obtained by *pointing* (51, 55).

146. In (198) if the Int. be paid half-yearly, then $R = 1 + \frac{1}{2}r$, and the No. of payments is $2n$; $\therefore M = P(1+\frac{1}{2}r)^{2n}$: or, if it be paid every m^{th} part of a year, $M = P\left(1+\dfrac{r}{m}\right)^{mn}$, which $= Pe^{nr}$ [117], if m be supposed infinitely great, or the Int. paid every moment.

147. *To find the Amount of an annuity* (A) *for* n *years,* (i) *at Simple,* (ii) *at Compound Interest.*

(i) The 1st Ann. bears int. for $(n-1)$ yrs; $\therefore M = A\{1+r(n-1)\}$;

2nd	. . . . . . $(n-2)$	. .	$= A\{1+r(n-2)\}$;
&c.	. . . . &c.	. .	$=$ &c.
nth	 0	. .	$= A$;

$\therefore$ whole amount $= nA + \{1+2+\text{\&c.}+(n-1)\}rA = nA + \frac{1}{2}n(n-1)rA$.

(ii) The whole amount $= A(1+R+\text{\&c.}+R^{n-1}) = A\,.\,\dfrac{R^n-1}{R-1}$.

148. In practice, it is most common to reckon as the *present value* of an Annuity that sum, whose improved value for the given time would equal the amount of the Annuity at the end of it.

If this be V, then (i) $V(1+nr) = nA + \frac{1}{2}n(n-1)rA$;

(ii) $VR^n = A.\dfrac{R^n-1}{R-1}$, or $V = A\,.\,\dfrac{1-R^{-n}}{R-1} = \dfrac{A}{r}(1-R^{-n})$.

Hence, if P be the fine on renewal of a lease for n years, it may be converted into rent: for if A be the additional rent, then

$$P = \frac{A}{r}\{1-R^{-n}\}, \text{ and } A = \frac{Pr}{1-R^{-n}}.$$

If the annuity be *perpetual*, (as, for instance, if it be the rent of a freehold estate) then, putting $n = \infty$, we have $V = \dfrac{A}{r}$.

If it be a *reversionary* annuity, that is, one to begin after a certain number of years, m suppose, then $V = V_n - V_m = \dfrac{A}{r} . \{R^{-m} - R^{-n}\}$,

or for a perpetuity, $= \dfrac{A}{r} - V_m = \dfrac{A}{r} . R^{-m} = \dfrac{A}{r} . \dfrac{1}{(1+r)^m}$.

149. But now suppose we estimate it otherwise, as follows, by calculating the present value of each Annuity as it becomes payable. Thus we have

V_1 = present value of 1st Annuity = (i) $\dfrac{A}{1+r}$, or (ii) $\dfrac{A}{R}$,

V_2 = ... 2nd . . . $\dfrac{A}{1+2r}$, . . $\dfrac{A}{R^2}$,

&c. . . . . &c. . . . . &c. . . &c.

$\therefore$ V = present value of whole Annuity = $V_1 + V_2 +$ &c.

$$=\text{(i) } A\left\{\frac{1}{1+r} + \frac{1}{1+2r} + \&\text{c.} + \frac{1}{1+nr}\right\}, \text{ or (ii) } \frac{A}{R^n} . \frac{R^n - 1}{R-1} = \frac{A}{r}(1 - R^{-n}).$$

It appears then that at *Compound* Interest, (which is the case usually met with in actual practice) the present value of an Annuity, on whichever hypothesis we calculate it, would be the same; but not so, at *Simple* Interest. It is easy to shew the reason of this, and, in fact, to see that the latter is *not* a correct mode of estimating the Present Value of the Annuity.

For, V_1 (*Simple Int.*) is indeed the present value of a sum P, payable at the end of a year, but *not* if afterwards P is to be *continued* at interest for any time; since, though V_1 + its int. for a year = P, yet as the Int. is not added to the principal, it would still be only V_1 that was bearing Int. in the one case, whereas it would be P in the other, that is, in the actual case of the Annuity. Of course, this is not the case at Comp. Int., in which at the year's end V_1 with its Int., or P, becomes the principal for the second year.

150. *Equation of Payments.* A sum P is due at the end of time t, and P' at the end of time t'; to find the time at which both sums should be paid together, at Simple Interest.

Let x be the time; then the interest on P, which is paid after its time, should = the discount on P', which is paid before its time; or $Pr(x-t) = \frac{P'r(t'-x)}{1+r(t'-x)}$, from which quadratic x may be found.

In practice, however, it is usual to reckon the *interest* instead of discount in the latter case; when we shall have

$$P(x-t) = P'(t'-x), \text{ or } x(P+P') = Pt + P't'.$$

Generally, if there be n sums P_1, P_2, &c. due at the end of times t_1, t_2, &c., and if x be the equated time, on this latter supposition, for payment of the whole sum $P_1 + P_2 +$ &c. or $\Sigma(P)$, then the amount of $\Sigma(P)$ for time x ought to equal the sum of the amounts of the separate sums for their separate times, that is

$$\Sigma(P)\{1+rx\} = P_1(1+rt_1) + P_2(1+rt_2) + \&c. = \Sigma(P) + r\Sigma(Pt);$$

$$\therefore x.\Sigma(P) = \Sigma(Pt).$$

If, however, we wish to get the strict result in this case, we shall have the present value of $P_1 = \frac{P_1}{1+rt_1}$, of $P_2 = \frac{P_2}{1+rt_2}$, &c.: let S = sum of these; then S should = present value of $\Sigma(P)$, due at the end of x years: $\therefore \Sigma(P) = S(1+rx)$, which gives x.

151. In the solution of questions on Interest, Logarithms are often required, as in some of the following Examples.

Ex. 1. Find the amount of £100 in 50 years, at 5 per cent Comp. Int.

Here $M = PR^n$; $\therefore \log M = \log P + n \log R = \log 100 + 50 \log(1.05)$ $= 2 + 50 \times (.0211893) = 3.0594650$, (where we obtain log 1.05 from the Table in p. 64, by observing that $105 = 3 \times 5 \times 7$,) which being log 1146.74, we have $M =$ £1146 14*s* 10*d* nearly.

Ex. 2. In how many years will a sum of money increase m-fold at Comp. Int.?

Here $M = mP = PR^n$;

$$\therefore m = R^n, \ \log m = n \log R, \text{ and } n = \log m \div \log R.$$

Thus to find when a sum will double itself at 5 per cent:

here $m = 2$, $R = 1.05$; and $n = \frac{\log 2}{\log 1.05} = \frac{.3010300}{.0211893} = 14.2$ years.

Ex. 3. How many years' purchase should be given for an estate, money making 5 per cent.?

Here [147] $V = \frac{A}{r} = \frac{A}{.05} = 20A$; that is, the estate is worth 20 times the rent, or, as it is said, is worth 20 years' purchase.

Ex. 28.

1. In what time at 5 per cent Comp. Int. will £100 amount to £1000?
2. What will £50 amount to at 5 per cent Comp. Int., in 10 yrs, interest paid half-yearly?
   (Given $\log 8193 = 3.9134430$, $\log 8.1931 = .9134483$.)
3. How long will a sum take to double itself at 4 per cent, (i) at Simp. Int., (ii) at Comp. Int.?
4. What is the amount of a farthing at 6 per cent Comp. Int. for 200 yrs? (Given $\log 1.199247 = .0789088$, nearly.)
5. *A* leaves *B* £1000 a year to accumulate for 3 years at 4 per cent Comp. Int.: what sum will *B* have to receive?
6. Find the present value of the above legacy.
7. How many years' purchase should be given for a freehold estate at 4 per cent, Comp. Int.?
8. What is the present value at 5 per cent of an estate of £1000 a year, (i) to be entered on immediately, (ii) after 3 years?
9. The reversion of a freehold estate of £882 per annum, to commence two years hence, is to be sold: find its present value at 5 per cent, Comp. Int.
10. A banker borrows at $3\frac{1}{2}$ per cent, interest payable yearly: he lends at 5 per cent, interest payable quarterly, and gains thus £441 in a year: how much does he borrow?
11. The rent of a farm is £A, and a fine of £P is required for a lease of n years: what ought to be the fine for one of $(m+n)$ years?
12. An annuity of £1000 for 4 years is left between *A* and *B* in the proportion of their ages 25 and 16, and they arrange accordingly that *A* shall have it the first two years, and *B* the other two: find the present value of each legacy at Comp. Int.

CHAPTER IX.

CONTINUED FRACTIONS.

152. Every expression of the form $a \pm \cfrac{b}{c \pm \cfrac{d}{c \pm \&c.}}$, or, as we shall write it, $a \pm \frac{b}{c \pm}\,\frac{d}{e \pm \&c.}$, is called a *Continued Fraction*, and is said to be *rational* or *irrational*, as the n° of its terms is *finite* or *infinite*.

The Fractions, however, of this kind with which we have commonly to do, and to which the following remarks will be restricted, are of the form $a + \cfrac{1}{b + \cfrac{1}{c + \&c.}}$ or $a + \frac{1}{b +}\,\frac{1}{c + \&c.}$, where a, b, c, &c. are all *positive numerical integers*.

153. *To convert any given fraction to a continued fraction.*

Let $\frac{m}{n}$ be the given fraction: divide m by n, with quotient a and rem^r p, n by p with quotient b and rem^r q, and so on, as in the process for finding the G.C.M. of m and n:

then $\frac{m}{n} = a + \frac{p}{n}$, $\frac{n}{p} = b + \frac{q}{p}$, $\frac{p}{q} = c + \frac{r}{q}$, &c. and $\frac{m}{n} = a + \frac{1}{b +}\,\frac{1}{c + \&c.}$, where, of course, if m be less than n, the first quotient a will be *zero*.

Now since, by following the process for the G.C.M., we must always come at last to a rem^r *zero*, it follows that any commensurable fraction may always be converted to a *terminating*, or, as it has been called above, a *rational* continued fraction.

Ex. Reduce $\frac{1051}{329}$ and $\frac{329}{1051}$ to continued fractions.

```
329) 1051 (3
      987
      ---
       64) 329 (5
           320
           ---
             9) 64 (7
                63
                --
                 1) 9 (9
```

Here (i) $\frac{1051}{329} = 3 + \cfrac{1}{5 + \cfrac{1}{7 + \cfrac{1}{9}}} = 3 + \frac{1}{5 +}\,\frac{1}{7 +}\,\frac{1}{9}$;

(ii) $\frac{329}{1051} = \frac{1}{3 +}\,\frac{1}{5 +}\,\frac{1}{7 +}\,\frac{1}{9}$, the first quotient being zero.

Ex. 29.

Reduce to continued fractions

1. $\frac{43}{30}$. 2. $\frac{157}{68}$. 3. $\frac{130}{421}$. 4. $\frac{189}{55}$. 5. $\frac{67}{88}$. 6. $\frac{159}{47}$. 7. $\frac{47}{257}$. 8. $\frac{457}{204}$.
9. $\frac{406}{391}$. 10. $\frac{1702}{1380}$. 11. $\frac{2065}{4626}$. 12. $\frac{2091}{568}$.

154. *To convert a quadratic surd to a continued fraction.*

A surd cannot, of course, be expressed as a *terminating* continued fraction; since then, by the mere labour of performing the operations, this might be reduced to the form of a common fraction, or the surd be expressed as a commensurable quantity. The method, however, of converting a quadratic surd to an *irrational* continued fraction, may be best seen by the following example, the steps of which are explained below.

Ex. $\sqrt{7}=2+(\sqrt{7}-2)=2+\dfrac{3}{\sqrt{7}+2}$, $\dfrac{\sqrt{7}+2}{3}=1+\dfrac{\sqrt{7}-1}{3}=1+\dfrac{2}{\sqrt{7}+1}$,

$\dfrac{\sqrt{7}+1}{2}=1+\dfrac{\sqrt{7}-1}{2}=1+\dfrac{3}{\sqrt{7}+1}$, $\dfrac{\sqrt{7}+1}{3}=1+\dfrac{\sqrt{7}-2}{3}=1+\dfrac{1}{\sqrt{7}+2}$,

$\dfrac{\sqrt{7}+2}{1}=\sqrt{7}+2=4+\dfrac{3}{\sqrt{7}+2}$, by taking the value of $\sqrt{7}$ from the first step, after which the same quotients will be repeated;

$$\therefore \sqrt{7}=2+\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{4+}\;\frac{1}{1+\&c.}$$

[*Explanation.* In the first step, 2 is set down as the greatest integer in $\sqrt{7}$, and then the quantity $\sqrt{7}-2$ is both mult. and div. by $\sqrt{7}+2$; in the second step, $\dfrac{\sqrt{7}+2}{3}$ is the reciprocal of $\dfrac{3}{\sqrt{7}+2}$, just obtained, 1 is the greatest int. in $\dfrac{\sqrt{7}+2}{3}$, which $=1+\left(\dfrac{\sqrt{7}+2}{3}-1\right)$ $=1+\dfrac{\sqrt{7}-1}{3}$; and then $\dfrac{\sqrt{7}-1}{3}$ is both mult. and div. by $\sqrt{7}+1$; and so on.]

It will be observed that the quotients begin to recur, as soon as we have come to a quotient double of the first: this, we shall shew hereafter, will always be the case.

Ex. 30.

Express as continued fractions

1. $\sqrt{5}$. 2. $\sqrt{6}$. 3. $\sqrt{8}$. 4. $\sqrt{11}$. 5. $\sqrt{12}$. 6. $\sqrt{13}$.
7. $\sqrt{19}$. 8. $\sqrt{21}$. 9. $\sqrt{22}$. 10. $\sqrt{23}$. 11. $\sqrt{33}$. 12. $\sqrt{55}$.

---

155. We shall now explain certain remarkable properties of the continued fraction $x=a+\dfrac{1}{b+}\;\dfrac{1}{c+\&c.}$.

The fractions formed by taking one, two, three, &c. of the quotients a, b, c, &c.

$$\left(\text{as } a = \frac{a}{1},\ a + \frac{1}{b} = \frac{ab+1}{b},\ a + \cfrac{1}{b + \cfrac{1}{c}} = a + \frac{c}{bc+1} = \frac{abc+a+c}{bc+1},\ \&\text{c.}\right),$$

are called *converging* fractions, because, as will be seen, they approach more and more nearly to the value of x. At present, it is at once plain that they are alternately less and greater than the true value of x: thus a is too small, $a + \frac{1}{b}$ is too great, because part of the denr is omitted, $a + \cfrac{1}{b + \cfrac{1}{c}}$ is too small, because $b + \frac{1}{c}$ is too great; and so on.

If x be less than 1, we shall have $a = 0$, and the first convergent will be $\frac{0}{1}$. If we begin to reckon with $\frac{a}{1}$ or $\frac{0}{1}$ in all cases, we may state the above Law as follows: The convergents of an odd order are all *less* and of an even order *greater* than the true value of the fraction.

156. *From a given continued fraction to obtain the corresponding series of converging fractions.*

If we consider the first three convergents in [155], we shall see that the numr of the *third* fraction $abc + a + c$, $= c(ab+1) + a$, that is, it may be formed by mult. the numr of the *second* by the third quotient, and adding in the numr of the *first*: and similarly for the denr.

Let then $\frac{p}{q}, \frac{p'}{q'}, \frac{p''}{q''}$, be any three consecutive convergents, with quotients m, m', m'', and suppose the above Law to hold good for the last of these, so that $p'' = m''p' + p$, $q'' = m''q' + q$. Now $\frac{p'''}{q'''}$ differs from $\frac{p''}{q''}$ only in taking in another quotient m''', and may therefore be obtained from $\frac{p''}{q''}$ by merely writing in it $m'' + \frac{1}{m'''}$ for m''; hence

we have $$\frac{p'''}{q'''} = \frac{\left(m'' + \frac{1}{m'''}\right)p' + p}{\left(m'' + \frac{1}{m'''}\right)q' + q} = \frac{m'''(m''p' + p) + p'}{m'''(m''q' + q) + q'} = \frac{m'''p'' + p'}{m'''q'' + q'},$$

that is, the Law still holds good; and, having been shewn to be true for the *third* convergent, it is therefore generally true.

Ex. 1. In [156 Ex. i.] we have quotients 3, 5, 7, 9, convergents $\frac{3}{1}$, $\frac{16}{5}$, $\frac{115}{36}$, $\frac{1051}{293}$; where the first two are formed from 3, $3+\frac{1}{5}$; and then the third $=\dfrac{7\times 16+3}{7\times 5+1}$, and the fourth $=\dfrac{9\times 115+16}{9\times 36+5}$.

In (ii) we have quotients 0, 3, 5, 7, 9, convergents $\frac{0}{1}$, $\frac{1}{3}$, $\frac{5}{16}$, $\frac{36}{115}$, $\frac{329}{1051}$.

We might have set $\frac{1}{0}$ in each case as the *first* convergent, (that is, before $\frac{3}{1}$ or $\frac{0}{1}$ respectively,) since it would be found that the *third* convergent would still be formed from the *first* and *second*, according to the Law above stated, upon which fact alone the reasoning of this Art. depends. We shall not, however, give $\frac{1}{0}$ or $\frac{0}{1}$ in the Ansrs.

Ex. 2. In [154 Ex.] we have quotients 2, 1, 1, 1, 4, 1, 1, 1, &c. convergents $\frac{1}{0}$, $\frac{2}{1}$, $\frac{3}{1}$, $\frac{5}{2}$, $\frac{8}{3}$, $\frac{37}{14}$, $\frac{45}{17}$, $\frac{82}{31}$, $\frac{127}{48}$, &c., which fractions are all nearer and nearer approximations to the value of $\sqrt{7}$, and alternately greater and less than it.

Ex. 31.

Obtain the converging fractions to the value of

1. $\frac{64}{265}$. 2. $\frac{285}{127}$. 3. $\frac{471}{972}$. 4. $\frac{743}{611}$. 5. $\frac{2431}{1051}$. 6. $\frac{1028}{9119}$.

Obtain the convergents, with only two figures in the denr, which approach nearest to the value of

7. $\sqrt{10}$. 8. $\sqrt{14}$. 9. $\sqrt{15}$. 10. $\sqrt{17}$. 11. $\sqrt{18}$. 12. $\sqrt{20}$.

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157. *If* $\frac{\mathrm{p}}{\mathrm{q}}$, $\frac{\mathrm{p}'}{\mathrm{q}'}$, *be consecutive convergents, then* $\frac{\mathrm{p}}{\mathrm{q}} \sim \frac{\mathrm{p}'}{\mathrm{q}'} = \frac{1}{\mathrm{qq}'}$, *or* $\mathrm{pq}' \sim \mathrm{p}'\mathrm{q} = 1$.

We see that this Law holds with the first two fractions, $\frac{a}{1}$, $\frac{ab+1}{b}$:

assume that it holds with $\frac{p}{q}$, $\frac{p'}{q'}$;

then $p''q' \sim p'q'' = (m''p'+p)\,q' \sim p'\,(m''q'+q) = pq' \sim p'q = 1$;
that is, the Law still holds, and will therefore be generally true.

In forming a series of converging fractions, this is a useful test to try the accuracy of the work, being so very easy of application.

Thus in [156 Ex. 2]

$$\frac{3}{1}-\frac{5}{2}=\frac{1}{1.2},\ \frac{5}{2}-\frac{8}{3}=\frac{-1}{2.3},\ \frac{8}{3}-\frac{37}{14}=\frac{1}{3.14},\ \frac{37}{14}-\frac{45}{17}=\frac{-1}{14.17},\ \&c.$$

Cor. 1. Hence every converging fraction $\frac{p}{q}$ is in its lowest terms: for if p and q had any common measure, it would also (63) be a measure of $pq' \sim p'q$ or unity.

Cor. 2. Hence we may always (when possible) find one solution of the Indeterminate Equation of the first degree, $ax+by=c$.

For convert $\frac{a}{b}$ to a continued fraction, and let $\frac{p}{q}$ be the convergent last but one; then $aq \sim bp = 1$, or $acq \sim bcp = c$, so that $x = \pm cq$, $y = \mp cp$, is a solution, and may be used for α and β in (138).

Ex. $18x + 7y = 127$.

7) 18 (2
14
4) 7 (1
4
3) 4 (1
3
1) 3 (3

Here we have quotients 2, 1, 1, 3, convergents $\frac{1}{0}$, $\frac{2}{1}$, $\frac{3}{1}$, $\frac{5}{2}$, $\frac{18}{7}$, and $18.2 - 7.5 = 1$: hence $18\,(254) - 7\,(635) = 127 = 18x + 7y$;

$$\therefore\ 18\,(254 - x) = 7\,(y + 635),$$

$$\left.\begin{array}{l} \text{and } 254 - x = 7t, \text{ or } x = 254 - 7t = 2 \\ \quad y + 635 = 18t, \quad y = 18t - 635 = 13 \end{array}\right\} \text{ if } t = 36.$$

The above is a useful method, whenever the value of c is *small*: otherwise, as in the above instance, it leads to large numbers.

158. *The successive convergents approach more and more nearly to the true value of the continued fraction.*

Let $\frac{p}{q}, \frac{p'}{q'}, \frac{p''}{q''}$, be consecutive convergents to the value of x.

Now x differs from $\frac{p''}{q''}$ only by taking instead of m'' the *complete quotient* $m'' + \frac{1}{\&c.} = M$ suppose, which is always greater than unity;

$$\therefore \text{ since } \frac{p''}{q''} = \frac{m''p' + p}{m''q' + q}, \text{ we have } x = \frac{Mp' + p}{Mq' + q},$$

$$\text{and } \frac{p}{q} \sim x = \frac{M\,(pq' \sim p'q)}{q\,(Mq' + q)} = \frac{M}{q\,(Mq' + q)},\quad x \sim \frac{p'}{q'} = \frac{1}{q'\,(Mq' + q)}.$$

Now $M > 1$, and $q < q'$; $\therefore$ on both accounts, $\frac{p}{q} \sim x > x \sim \frac{p'}{q'}$, that is, $\frac{p'}{q'}$ is nearer to x than $\frac{p}{q}$ is, one being $>$ [155] and the other $<$ than it.

Cor. Hence also it follows that $\frac{p}{q} \times \frac{p'}{q'} \gtrless x^2$, according as $\frac{p}{q} \gtrless x$:

$$\text{for, (155) if } \frac{p}{q} > x, \text{ then } \frac{p}{q} : x > x : \frac{p'}{q'}, \text{ or } \frac{pp'}{qq'} > x^2;$$

$$\text{and, if } \frac{p}{q} < x, \text{ then } \frac{p}{q} : x < x : \frac{p'}{q'}, \text{ or } \frac{pp'}{qq'} < x^2.$$

159. *The error made in taking* $\frac{p}{q}$ *for* x *lies between* $\frac{1}{qq'}$ *and* $\frac{1}{q(q+q')}$

$$\text{For [158] } \frac{p}{q} \sim x = \frac{M}{q(Mq'+q)} = \frac{1}{q\left(q' + \frac{q}{M}\right)}, \text{ and is } \therefore < \frac{1}{qq'} \text{ but} > \frac{1}{q(q'+q)}.$$

COR. 1. Since $q<q'$, we have also, *a fortiori*, the error $<\frac{1}{q^2}, > \frac{1}{2q'^2}$, which limits, on account of their simplicity, are preferable in practice.

COR 2. If we wish to find a convergent $\frac{p}{q}$, differing from x in value by a quantity less than any given quantity $1 \div a$, we must continue the series until we come to one, whose denʳ $q \overline{\gtrless} \sqrt{a}$.

Ex. In [156 Ex. 2] $\frac{82}{31} = 2.645161$, being one of the *even* fractions is *less* than $\sqrt{7}$; and the error made in taking it for the true value is $< \frac{1}{(31)^2}, > \frac{1}{2(48)^2}$, that is $< \frac{1}{961}, > \frac{1}{4608}$, and, *a fortiori*, $< \frac{1}{900}$ or $.00\dot{1}$, $> \frac{1}{5000}$ or 0002, from which we see that it will not affect the third place of decimals, so that $\sqrt{7} = 2.645$.

160. *Those convergents which immediately precede large quotients are near approximations to the true value of the continued fraction.*

For [157] $\frac{p}{q} \sim \frac{p'}{q'} = \frac{1}{qq'}$, $\frac{p''}{q''} \sim \frac{p'}{q'} = \frac{1}{q'q''}$: and $q'' = m''q' + q$, and is therefore much greater than q if m'' be large, that is, $\frac{1}{q'q''}$ is much less than $\frac{1}{qq'}$, or $\frac{p'}{q'}$ differs much less from $\frac{p''}{q''}$ than it does from $\frac{p}{q}$; and therefore x, which lies between $\frac{p'}{q'}$ and $\frac{p''}{q''}$ and also between $\frac{p'}{q'}$ and $\frac{p}{q}$, $\left(\frac{p''}{q''}\right.$ and $\frac{p}{q}$ being *both* greater, or *both* less, than $\frac{p'}{q'}$ and $\left.x,\right)$ must be much nearer to $\frac{p'}{q'}$ than to $\frac{p}{q}$.

161. *Any convergent* $\frac{\mathrm{p}}{\mathrm{q}}$ *approaches more nearly to the value of* x *than any other fraction whatsoever, whose den*ʳ *is less than* q.

Let $\frac{r}{s}$ be a fraction intermediate in value between $\frac{p}{q}$ and x: if this fraction be one of the convergents, the proposition is already proved by [158]: but, if not, since $\frac{r}{s}$ lies between $\frac{p}{q}$ and x, it lies, *a fortiori*, between $\frac{p}{q}$ and $\frac{p'}{q'}$, since x lies between them: hence $\frac{r}{s} \sim \frac{p'}{q'} < \frac{p}{q} \sim \frac{p'}{q'} < \frac{1}{qq'}$, or $rq' \sim p's < \frac{s}{q}$, which is impossible if s be *less* than q, since r, s, p', q' are all integers.

162. The following examples of the use of Continued Fractions are also noticeable.

Ex. 1. To approximate to the roots of $x^2 - 5x - 3 = 0$ by Continued Fractions.

Here $x = \frac{1}{2}(5 \pm \sqrt{37})$: taking first the upper sign, we may proceed as in [154],

$$\frac{\sqrt{37}+5}{2} = 5 + \frac{\sqrt{37}-5}{2} \quad 5 + \frac{6}{\sqrt{37}+5}, \text{ \&c.},$$

and shall find the quotients 5, 1, 1, 5, 1, 1, &c., and $x = \frac{5}{1}, \frac{6}{1}, \frac{11}{2}, \frac{61}{11}, \frac{72}{13}$, &c. Again, taking the lower sign, $-x = \frac{1}{2}(\sqrt{37}-5)$, with which we may proceed as before, $\frac{\sqrt{37}-5}{2} = 0 + \frac{6}{\sqrt{37}+5}$, &c., getting the same quotients (except the first) as above, so that $-x = 0, \frac{1}{1}, \frac{1}{2}, \frac{6}{11}, \frac{7}{13}$, &c.

Ex. 2. *To solve the equation $2^x = 3$ by Continued Fractions.*

Find in such a case the integer next lower than x, that is, in the case before us, 1; put $x = 1 + \frac{1}{y}$; then $2.2^{\frac{1}{y}} = 3$, or $\left(\frac{3}{2}\right)^y = 2$: find now the integer next lower than y, which is in this case, also 1; put $y = 1 + \frac{1}{z}$; then $\frac{3}{2}\left(\frac{3}{2}\right)^{\frac{1}{z}} = 2$, or $\left(\frac{4}{3}\right)^z = \frac{3}{2}$: put $z = 1 + \frac{1}{v}$; then $\left(\frac{9}{8}\right)^v = \frac{4}{3}$: put $v = 2 + \frac{1}{w}$, &c.; then we have $x = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}$ &c. $= \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{8}{5}$, &c.; and the error made in taking $\frac{8}{5} = 1.6$ for x is $< \frac{1}{25} = .04$, or x lies between 1.56 and 1.6.

Ex. 3. (i) Let $x = \frac{1}{p+}\,\frac{1}{p+\text{\&c.}} = \frac{1}{p+x}$; $\therefore x^2 + px - 1 = 0$, and $x = -\frac{1}{2}p + \frac{1}{2}\sqrt{(p^2+4)}$: hence $\frac{1}{2}\sqrt{(p^2+4)} = x + \frac{1}{2}p = \frac{1}{2}p + \frac{1}{p+}\,\frac{1}{p+\text{\&c.}}$

Put $p = 2$; then $\sqrt{2} = 1 + \frac{1}{2+}\,\frac{1}{2+\text{\&c.}}$; and so in other cases.

(ii) Let $x = \frac{1}{p+}\,\frac{1}{q+}\,\frac{1}{p+}\,\frac{1}{q+\text{\&c.}} = \frac{1}{p+}\,\frac{1}{q+x}$;

then, as in Ex. 2, $x = -\frac{1}{2}q + \frac{1}{2p}\sqrt{(p^2q^2 + 4pq)}$:

Put $p = 2$, $q = 3$; then $\sqrt{15} = 3 + 2\left\{\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+\text{\&c.}}\right\}$; and so on.

Ex. 4. Let $x = \sqrt{[a + \sqrt{\{a + \sqrt{(a + \text{\&c.})}\}}]} = \sqrt{(a+x)}$; $\therefore x = \frac{1}{2} + \sqrt{(a + \frac{1}{4})}$.

Put $a = 2$; then $\sqrt{[2 + \sqrt{\{2 + \sqrt{(2 + \text{\&c.})}\}}]} = 2$; and so on.

Ex. 32.

1. The ratio of the circumference of a circle to its diameter being 3.14159 : 1, shew that this is nearly that of 22 : 7, more nearly that of 333 : 106, and still more nearly that of 355 : 113. Find to how many places of decimals each of these may be depended on as agreeing with the true value.

2. Two clergymen having to make up together the income of a common schoolmaster, and the net values of their livings being £297 and £685, shew that they should pay nearly in the ratio of 10 : 23.

3. The lb. Troy weighs 22.8157 inches of distilled water, the lb. Av. 27.7274 inches: shew that the two lbs. are nearly in the ratio of 96 : 79.

4. The height of the great Pyramid being 479 ft. and of St. Paul's 404 ft., shew that they are nearly in the ratio of 13 : 11.

5. Find the value of $1 + \frac{1}{3+}\,\frac{1}{5 + \&c.}$ accurately to within .0001.

6. Shew that the ratio of the diagonal of a square to the side may be nearly expressed by 99 : 70, and that of the diagonal of a cube to its side by 97 : 56. What is the Limit of the error made in taking these fractions for the respective ratios?

7. Mercury makes its revolution in 88 days, Venus in 225 days: shew that, in the time taken by Venus to make 9 complete revolutions, Mercury will also have made very nearly a certain number of complete revolutions.

8. Two scales whose zero points coincide are placed side by side, and the space between consecutive divisions in the one is to that in the other as 1 : 1.06577; shew that the divisions which most nearly coincide are 15 and 16, 61 and 65, 76 and 81, &c.

9. Find x correct to two places of decimals in the equations,

$$\text{(i) } 3x^2 = 7, \quad \text{(ii) } 3x^2 - 5x + 1 = 0.$$

10. Find x by an equation when

$$\text{(i) } x = p + \frac{1}{q+}\,\frac{1}{r+}\,\frac{1}{q + \&c.}, \quad \text{(ii) } x = p + \frac{1}{q+}\,\frac{1}{r+}\,\frac{1}{s+}\,\frac{1}{r + \&c.}.$$

11. Find x correct to two places of decimals in the equations,
(i) $x = \sqrt{[3 + \sqrt{\{3 + \sqrt{(3 + \&c.)}\}}]}$. (ii) $3^x = \frac{1}{2}$. (iii) $2^x = \frac{2}{3}$.

12. Find x by an equation when

$$x = \sqrt{\left\{a + \frac{b}{\sqrt{\left(a + \frac{b}{\sqrt{(a + \&c.)}}\right)}}\right\}}.$$

163. Let a be the greatest integer in $\sqrt{N}$ [154]: then $\sqrt{N}=a+(\sqrt{N}-a)$ $=a+\dfrac{r}{\sqrt{N}+a}$, if $r=N-a^2$, where $\dfrac{r}{\sqrt{N}+a}<1$, since $\sqrt{N}-a<1$; so

$\dfrac{\sqrt{N}+a}{r}=b+\dfrac{\sqrt{N}+a-rb}{r}=b+\dfrac{r'}{\sqrt{N}+a'}$, if $a'=rb-a$, $r'=\dfrac{N-a'^2}{r}$,

$\dfrac{\sqrt{N}+a'}{r'}=b'+\dfrac{\sqrt{N}+a'-r'b'}{r'}=b'+\dfrac{r''}{\sqrt{N}+a''}$, if $a''=r'b'-a'$, $r''=\dfrac{N-a''^2}{r'}$,

(the quotients b, b', &c. being all *positive integers*, since the fractions $\dfrac{\sqrt{N}+a}{r}$, $\dfrac{\sqrt{N}+a'}{r'}$, &c. are all >1,) and so on, until, as will be shewn, we shall at last arrive at a den^r *unity*, which gives the last quotient in each period.

The above formulæ, $a'=rb-a$, $rr'=N-a'^2$ might be applied to find a', r' from a, b, r, and then a'', r'', from a', b', r', &c., without going through the actual calculations in any case: but the method itself in [154] is very easy of application; and the above is rather given by way of introduction to the following remarks upon it.

164. (i) *The quantities* a′, a″, *&c.* r′, r″, *&c. are all positive integers.*

For let $\dfrac{p}{q}$, $\dfrac{p'}{q'}$, $\dfrac{p''}{q''}$, be consecutive convergents to $\sqrt{N}$, corresponding to the quotients b, b', b''; then, since instead of b'', the *complete* quotient is $\dfrac{\sqrt{N}+a''}{r''}$, we have, as in [158],

$$\sqrt{N}=\frac{\dfrac{\sqrt{N}+a''}{r''}p'+p}{\dfrac{\sqrt{N}+a''}{r''}q'+q}=\frac{(\sqrt{N}+a'')p'+r''p}{(\sqrt{N}+a'')q'+r''q};$$

whence, equating rational and irrational parts, we have

$$a''p'+r''p=q'N,\quad\text{or } a''(pq'-qp')=pp'-qq'N,$$
$$a''q'+r''q=p',\qquad r''(pq'-qp')=q^2N-p^2:$$

but $pq'-qp'=+1$ or -1, according as $\dfrac{p}{q}\gtrless\dfrac{p'}{q'}$, and $\therefore$ according as $\sqrt{N}\gtrless\dfrac{p'}{q'}$ or $q'^2N\gtrless p'^2$, and also [158 Cor.] as $\dfrac{pp'}{qq'}\gtrless N$ or $pp'\gtrless qq'N$: hence it appears that a'' and r'' are *positive integers*, and similarly for any other pair of these quantities, except the first two pairs, which are *assumed* to be positive integers in the above proof, (in which we take for granted the formation of the first two convergents,)

and for these we may take (0, 1) (a, r), respectively, corresponding to $\sqrt{N} = \dfrac{\sqrt{N}+0}{1}, \dfrac{\sqrt{N}+a}{r}$.

(ii) *The extreme limit of the value of* a′, a″, *&c. is* a.

For $r'r'' = N - a''^2$, and $r'\ r''$ are positive integers; $\therefore \sqrt{N} > a''$: but a is the greatest integer in $\sqrt{N}$; $\therefore$ the limit of a'' is a.

(iii) *The extreme limit of the value of* r, r′, *&c. or of* b, b′, *&c. is* 2a.

For $a' + a'' = r'b'$, and r' and b' are positive integers, and by (ii) $a' + a''$ cannot exceed $2a$; $\therefore$ neither r' nor b' can exceed $2a$.

Cor. Hence, since the values of a'' and r'' are limited respectively to a and $2a$, the *same* values of a'' and r'' must recur together within $2a^2$ terms, that is, the series of quotients must recur.

(iv) *If any quotient* b′ = 2a, *then* r′ = 1, *and* a′ = a″ = a, as appears plainly from the reasoning in (iii).

165. *The recurrence begins with the first complete quotient.*

Let the quotients be b_1, b_2, &c. corresponding to a_1, a_2, &c. r_1, r_2, &c., so that

$$\sqrt{N} = a + \frac{r}{\sqrt{N}+a} = a_1 + \frac{r_1}{\sqrt{N}+a_1}, \quad \frac{\sqrt{N}+a_1}{r_1} = b_1 + \frac{r_2}{\sqrt{N}+a_2}, \text{ \&c.}:$$

and let the quantities a_m, r_m, recur, so that, after $n-m$ terms more, we obtain $a_n = a_m$, $r_n = r_m$. Then, since

$$r_{m-1}r_m = N - a_m^2 = N - a_n^2 = r_{n-1}r_n,$$

we have also $r_{m-1} = r_{n-1}$.

Again, $a_{m-1}+a_m = b_{m-1}r_{m-1}$, and $a_{n-1}+a_n = b_{n-1}r_{n-1}$ or $a_{n-1}+a_m = b_{n-1}r_{m-1}$; hence we have $\dfrac{a_{m-1} \sim a_{n-1}}{r_{m-1}} = b_{m-1} \sim b_{n-1} =$, therefore, an *integer* or *zero*.

But [164] if $\dfrac{p}{q}, \dfrac{p'}{q'}, \dfrac{p''}{q''}$, be the three consecutive convergents to $\sqrt{N}$, corresponding to the quotients b_{m-3}, b_{m-2}, b_{m-1}, then we have

$$a_{m-1}q' + r_{m-1}q = p', \text{ or } a_{m-1} = \frac{p'}{q'} - \frac{q}{q'}r_{m-1} = \left(a + \frac{s}{q'}\right) - \frac{q}{q'}r_{m-1}, \text{ suppose,}$$

where s is positive and less than q'; for the *first* convergent a is less than $\sqrt{N}$, and, therefore, *any* convergent $\dfrac{p'}{q'}$, being nearer to a than $\sqrt{N}$, will necessarily exceed a by a proper fraction. Hence we have $a - a_{m-1} = \dfrac{q}{q'}r_{m-1} - \dfrac{s}{q'}$, which is $< r_{m-1}$, since $q < q'$;

and, in like manner, we obtain $a - a_{n-1} < r_{n-1} < r_{m-1}$: and, therefore, *a fortiori*, $a_{m-1} \sim a_{n-1} < r_{m-1}$, or $\frac{a_{m-1} \sim a_{n-1}}{r_{m-1}} < 1$, and, consequently, it must = *zero*, so that $a_{m-1} = a_{n-1}$, and $b_{m-1} = b_{n-1}$.

In like manner, $a_{m-2} = a_{n-2}$, $r_{m-2} = r_{n-2}$, $b_{m-2} = b_{n-2}$, &c., and so on, backwards, until we come to a_1 (or a) $= a_{n-m}$, r_1 (or r) $= r_{n-m}$, $b_1 = b_{n-m}$; that is, the recurrence will begin with the first complete quotient.

166. *The last integral quotient will be always* 2a.

For let the last complete quotient be $\frac{\sqrt{N} + a_m}{r_m}$: then, since this is followed by $\frac{\sqrt{N} + a_1}{r_1}$ or $\frac{\sqrt{N} + a}{r}$, we shall have $a + a_m = b_m r_m$ and $N - a^2 = r r_m$: but $r = N - a^2$; $\therefore\ r_m = 1$, and $b_m = a + a_m$.

But, reasoning as in (165), $a - a_m = \frac{q'}{q''} r_m - \frac{s'}{q''}$, (where $s' < q''$) $= \frac{q'}{q''} - \frac{s'}{q''}$, which cannot be satisfied, (since a and a_m are integers, and q' and s' are each less than q'',) except $q' = s'$, and $\therefore\ a_m = a$, or $b_m = 2a$.

167. If N be of the form (i) $a^2 + 1$, then $\sqrt{N} = a + \{\sqrt{(a^2 + 1)} - a\}$ $= a + \frac{1}{\sqrt{(a^2 + 1)} + a} = a + \frac{1}{2a +}\, \frac{1}{2a + \&c.}$, and the period will consist of one term only.

If N be of the form (ii) $a^2 - 1$, or (iii) $a^2 + a$, or (iv) $a^2 - a$, it will consist of *two* terms, as will be easily seen on trial, the quotients being

for (ii) $a - 1$, | 1, $2(a - 1)$ | &c., for (iii) a, | 2, $2a$, | &c.,
for (iv) $a - 1$, | 2, $2(a - 1)$, | &c.

168. *The equation* $x^2 - Ny^2 = 1$ *can always be solved in positive integers, where* N *represents any integer not a square.*

For [164 i] $p^2 - Nq^2 = \pm r' = \pm 1$ [164 iv], if $\frac{p}{q}$ correspond to the last quotient $2a$, + or − according as $\frac{p}{q} \gtrless \sqrt{N}$.

Now all the convergents in even places are $> \sqrt{N}$, and in odd places $< \sqrt{N}$: and, if the No. of quotients in the period be *even*, all the fractions, corresponding to $2a$, will be in even places; but if *odd*, then the second, fourth, &c., fractions, corresponding to $2a$, will

be in even places; and, therefore, it is always possible to find an infinite No. of solutions for $x^2 - Ny^2 = 1$, by converting $\sqrt{N}$ to a continued fraction.

COR. 1. $x^2 - Ny^2 = -1$, is always possible, in an infinite number of ways, when the number of quotients in the period is *odd*.

COR. 2. Hence $x^2 - Ny^2 = \pm z^2$ may be solved (subject to the above limitations) by first solving $p^2 - Nq^2 = \pm 1$, and putting $x = pz$, $y = qz$.

COR. 3. If a be found among the values of r, r', &c., then $x^2 - Ny^2 = \pm a$ may be solved in integers, subject to limitations as above, by means of the equation $p^2 - Nq^2 = \pm r'$.

Ex. $x^2 - 7y^2 = 1$.

Referring to [156 Ex. 2] we have the quotients 2, 1, 1, 1, 4, 1, 1, 1, 4, &c. convergents $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{8}{3}, \frac{37}{14}, \frac{45}{17}, \frac{82}{31}, \frac{127}{48}$, &c.; $\therefore$ (the n° of terms in the period being *even*) we have $(8)^2 - 7(3)^2 = 1$, $(82)^2 - 7(31)^2 = 1$, &c.:

$$\text{or } x = 8,\ 82,\ \&c., \quad y = 3,\ 31,\ \&c.$$

The equation $x^2 - 7y^2 = -1$ is impossible.

Ex. 2. (i) $x^2 - 7y^2 = -3$, (ii) $x^2 - 7y^2 = 2$.

It appears from [154] that the complete quotients for $\sqrt{7}$ are $\frac{\sqrt{7}+2}{3}, \frac{\sqrt{7}+1}{2}, \frac{\sqrt{7}+1}{3}, \frac{\sqrt{7}+2}{1}$, &c., so that 3 being found in the denrs of the first, third, fifth, &c. of these, and 2 in those of the second, sixth, &c., the equations above given are possible, and the solutions are (referring to the convergents in Ex. 1),

$$\text{for (i) } x = 2,\ 5,\ 37,\ 82,\ \&c., \quad y = 1,\ 2,\ 14,\ 31,\ \&c.,$$
$$\text{for (ii) } x = 3,\ 45,\ \&c. \qquad y = 1,\ 17,\ \&c.$$

Ex. 33.

1. $x^2 - 3y^2 = 1$. 2. $x^2 - 8y^2 = 1$. 3. $x^2 - 23y^2 = 2$. 4. $x^2 - 23y^2 + 7 = 0$.
5. $x^2 - 13y^2 + 1 = 0$. 6. $x^2 - 44y^2 + 8 = 0$. 7. $x^2 - 44y^2 = 4$. 8. $x^2 - 61y^2 = \pm 5$.

CHAPTER X.

INDETERMINATE COEFFICIENTS, PARTIAL FRACTIONS, AND SERIES.

169. In any series $a_0 + a_1x + a_2x^2 + \&c.$, any term a_nx^n may be made > the sum of all that follow it, by sufficiently diminishing x.

For let k be the greatest coeff. in the whole series; then the extreme case will be when the terms following a_nx^n are all *positive* and have their coeff^s^ each $= k$, in which case their sum becomes $kx^{n+1} + kx^{n+2} + \&c. = kx^{n+1}(1 + x + \&c.) = kx^{n+1} \div (1 - x)$, if x be taken < 1: now $a_nx^n > \dfrac{kx^{n+1}}{1-x}$, if $a_n > \dfrac{kx}{1-x}$, or if $x < \dfrac{a_n}{a_n + k}$.

Cor. If x be taken $< \dfrac{a_0}{a_0 + k}$, the *first* term will be > sum of all that follow it; and, since a_0 may here have any value whatever, we see that, by sufficiently diminishing x, any series $a_1x + a_2x^2 + \&c.$ may be made less than any assignable quantity whatever.

170. If $a + bx + cx^2 + \&c. = a' + b'x + c'x^2 + \&c.$ for all finite values of x whatever, we must have $a = a'$, $b = b'$, $c = c'$, &c.

For $a - a' = (b' - b)\,x + (c' - c)\,x^2 + \&c.$ for all values of x whatever; but if $a - a'$ be finite, we have seen that, by taking x of proper value, we may make the series $(b' - b)\,x + \&c. < a - a'$; which is contradictory: hence $a - a' = 0$, or $a = a'$; and then $bx + cx^2 + \&c. = b'x + c'x^2 + \&c.$, or $b + cx + \&c. = b' + c'x + \&c.$ for all values of x whatever; $\therefore b = b'$, and so on. This Proposition we have, in fact, assumed as almost self-evident, in [116].

Cor. If $A + Bx + Cx^2 + \&c. = 0$, for all values of x, then $A = 0$, $B = 0$, &c.

171. If $A + \alpha = B + \beta$, where A, B represent *constant* and α, β *variable* quantities, which may become each at the same time less than any assignable quantity, then $A = B$, $\alpha = \beta$: for, if not, let $A = B \pm C$; then $\alpha \sim \beta = C$, and therefore α, β cannot be both at the same time $< C$, contrary to the hypothesis. [This result is sometimes employed in proving a main Proposition in Trigonometry.]

172. The following are applications of the above.

Ex 1. To expand $\dfrac{a'+b'x}{1+ax+bx^2}$ in ascending powers of x.

Assume $\dfrac{a'+b'x}{1+ax+bx^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$

when A, B, C, &c., as yet unknown, are called *Indeterminate Coeff*^{s};

$$\begin{aligned} \text{then} \qquad a' + b'x &= A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c. \\ &\quad + aAx + aBx^2 + aCx^3 + aDx^4 + \&c. \\ &\quad + bAx^2 + bBx^3 + bCx^4 + \&c.\,; \end{aligned}$$

hence, equating coeff^s, we have $A=a'$, $B+aA=b'$, $C+aB+bA=0$, &c.
so that $A=a'$, $B=b'-aa'$, $C=-aB-bA=a^2a'-ab'-ba'$, &c.,

and $\dfrac{a'+b'x}{1+ax+bx^2} = a' + (b'-aa')\,x + (a^2a'-ab'-ba')\,x^2 + \&c.$

Ex. 2. Given $y = ax + bx^2 + cx^3 + \&c.$ to express x in terms of y.

Since, when $y=0$, we have also $x=0$, it is at once plain that the series required for x cannot contain any *constant* term A; for then when $y=0$, we should have $x=A$: assume then

$x = Ay + By^2 + Cy^3 + \&c.$; but $y = ax + bx^2 + cx^3 + \&c.$;

$$\begin{aligned} \therefore x &= Aax + Abx^2 + Acx^3 + \&c. \\ &\qquad + Ba^2x^2 + 2Babx^3 + \&c. \qquad \therefore 1 = Aa,\ 0 = Ab + Ba^2, \\ &\qquad\qquad + Ca^3x^3 + \&c. \qquad 0 = Ac + 2Bab + Ca^3,\ \&c. \end{aligned}$$

hence $A = \dfrac{1}{a}$, $B = -\dfrac{b}{a^3}$, $C = \dfrac{2b^2-ac}{a^5}$, &c. and $x = \dfrac{y}{a} - \dfrac{by^2}{a^3} + \&c.$

Or thus: since $y = ax + bx^2 + cx^3 + \&c.$, and $x = Ay + By^2 + Cy^3 + \&c.$

$$\begin{aligned} \therefore y &= aAy + aBy^2 + aCy^3 + \&c. \\ &\qquad + bA^2y^2 + 2bABy^3 + \&c. \quad \therefore 1 = aA,\ 0 = aB + bA^2, \\ &\qquad\qquad + cA^3y^3 + \&c. \qquad 0 = aC + 2bAB + cA^3,\ \&c. \end{aligned}$$

whence the values, as before, of A, B, C, &c.

The above is an instance of what is called *Reversion of Series.*

Ex. 3. Revert the series $y = ax + bx^3 + cx^5 + \&c.$

Here, since by changing the sign of x we should only change the *sign*, and not the *value*, of y, it is plain that x must be also of the form $x = Ay + By^3 + Cy^5 + \&c.$; in fact, if we assumed as in Ex. 2, we should find the alternate coeff^s each *zero*. Making this assumption, and proceeding as before, we shall obtain

$$x = \frac{y}{a} - \frac{by^3}{a^4} + \frac{(3b^2-ac)\,y^5}{a^7} - \&c.$$

Obs. If the series to be reverted in any case be of the form $y = a' + ax + bx^2 + \&c.$, we must put $y - a' = z$, and express x in terms of z and its powers.

Ex. 34.

Expand in ascending powers of x to five terms, by Ind. Coeff^{ts}.

1. $\dfrac{1}{2-3x}$.   2. $\dfrac{1+x}{2+3x}$.   3. $\dfrac{1-x}{1-x+x^2}$.   4. $\dfrac{1+2x-3x^2}{1-2x+3x^2}$.

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5. $y = x + x^2 + x^3 + \&c.$ 6. $y = x - 2x^2 + 3x^3 - \&c.$

7. $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \&c.$ 8. $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \&c.$

9. $y = x + \dfrac{x^2}{1.2} + \dfrac{x^3}{1.2.3} + \&c.$ 10. $y = x - \dfrac{x^3}{1.2.3} + \dfrac{x^5}{1.2.3.4.5} - \&c.$

11. $y = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{61}{5040}x^7 + \&c.$ 12. $y = x - \frac{1}{6}x^3 + \frac{1}{24}x^5 - \frac{61}{5040}x^7 + \&c.$

Ex. 4. To resolve $\dfrac{x+3}{(x-1)(x+2)}$ into separate, or *partial*, fractions.

Assume $\dfrac{x+3}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$; then $x+3 = A(x+2) + B(x-1)$:

$\therefore 1 = A + B,\ 3 = 2A - B,$ when $A = \frac{4}{3},\ B = -\frac{1}{3}$;

$$\text{and } \frac{x+3}{(x-1)(x+2)} = \frac{4}{3(x-1)} - \frac{1}{3(x+2)}.$$

Or thus: assuming as before, multiply by each factor of the den^r in turn; then

(i) $\dfrac{x+3}{x+2} = A + \dfrac{B(x-1)}{x+2}$, where put $x - 1 = 0$, or $x = 1$; then $A = \frac{4}{3}$;

(ii) $\dfrac{x+3}{x-1} = \dfrac{A(x+2)}{x-1} + B$, ... $x + 2 = 0$, or $x = -2$; then $B = -\frac{1}{3}$.

This second method, besides being more simple, enables us to find at once the num^r of any one of the partial fractions, without finding the others, by merely multiplying by its den^r, and then putting this multiplier $= 0$: or, without so much trouble, we may set down at once the partial fractions in such a case as follows.

Ex. 5. $\dfrac{2x+3}{x^2+x-2} = \dfrac{2x+3}{(x+2)(x-1)} = \dfrac{-4+3}{(x+2)(-3)} + \dfrac{2+3}{(1+2)(x-1)}$

$$= \frac{1}{3(x+2)} + \frac{5}{3(x-1)},$$

where to form the first fraction we put $x = -2$, derived from $x + 2 = 0$, and to form the second $x = 1$, derived from $x - 1 = 0$.

Ex. 6. Let $\frac{3x+2}{x(x+1)^3} = \frac{A}{x} + \frac{B}{(x+1)^3} + \frac{C}{(x+1)^2} + \frac{D}{x+1}$, which assumption we are obliged to make, because, when these fractions are added together, their common den^r will only be the same as that of the given fraction, and therefore it *may* have been composed of such as these:

then, as in Ex. 5, $\frac{3x+2}{x(x+1)^3} = \frac{2}{x} + \frac{(-3)+2}{(-1)(x+1)^3} + \frac{C}{(x+1)^2} + \frac{D}{x+1}$

$$= \frac{2}{x} + \frac{1}{(x+1)^3} + \frac{C}{(x+1)^2} + \frac{D}{x+1}:$$

and now, before we can find C and D, we must subtract from the other side $\frac{1}{(x+1)^3}$, which leaves $\frac{2x+2}{x(x+1)^3}$, where the num^r is now divisible by $x+1$: striking out this factor from num^r and den^r,

$$\frac{2}{x(x+1)^2} = \frac{2}{x} + \frac{C}{(x+1)^2} + \frac{D}{x+1} = \frac{2}{x} + \frac{2}{(-1)(x+1)^2} + \frac{D}{x+1} = \frac{2}{x} - \frac{2}{(x+1)^2} + \frac{D}{x+1};$$

again, subtracting $-\frac{2}{(x+1)^2}$ from the other side, we have $\frac{2+2x}{x(x+1)^2}$, where numerator and denominator are again both divisible by $x+1$:

$$\text{hence } \frac{2}{x(x+1)} = \frac{2}{x} + \frac{D}{x+1} = \frac{2}{x} + \frac{2}{(-1)(x+1)} = \frac{2}{x} - \frac{2}{x+1};$$

$$\text{and } \therefore \frac{3x+2}{x(x+1)^3} = \frac{2}{x} + \frac{1}{(x+1)^3} - \frac{2}{(x+1)^2} - \frac{2}{x+1}.$$

Perhaps, however, the simplest way of obtaining the partial fractions corresponding to a *power* in the den^r, is as follows.

$$\text{Let } \frac{3x+2}{x(x+1)^3} = \frac{A}{x} + \frac{P}{(x+1)^3} = \frac{2}{x} + \frac{P}{(x+1)^3} \text{ as in Ex. 5;}$$

$$\therefore \frac{P}{(x+1)^3} = \frac{3x+2}{x(x+1)^3} - \frac{2}{x} = -\frac{2x^3+6x^2+3x}{x(x+1)^3} = -\frac{2x^2+6x+3}{(x+1)^3}:$$

now put $x+1=z$, or $x=z-1$; $\therefore \frac{P}{(x+1)^3} = -\frac{2(z-1)^2+6(z-1)+3}{z^3}$

$$= -\frac{2z^2+2z-1}{z^3} = -\frac{2}{z} - \frac{2}{z^2} + \frac{1}{z^3} = -\frac{2}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}, \text{ as before.}$$

N.B. In all the above cases, it is supposed that the given fraction is a *proper* fraction, that is, with num^r of lower dimensions than the den^r; otherwise it must be turned into a *mixed* quantity, and the above method may then be applied to the resulting fraction.

Ex. 35.

Resolve into partial fractions:

1. $\dfrac{x+1}{(x+2)(x+3)}$. 2. $\dfrac{x+1}{x(x-2)}$. 3. $\dfrac{3x+2}{x^2-1}$. 4. $\dfrac{x-3}{x(x+1)^2}$. 5. $\dfrac{2x}{(x-1)^2(x+1)}$.

6. $\dfrac{x^2-x+1}{x^2(x+1)}$. 7. $\dfrac{2x^2+1}{x^2(x+1)}$. 8. $\dfrac{3x-1}{x^2(x+1)^2}$. 9. $\dfrac{1-x-x^2}{x^2(x-1)^3}$. 10. $\dfrac{x^2}{(x+1)^2(x-1)^2}$.

Ex. 7. Resolve $\dfrac{3x^2+5x+1}{x(x^2+x+1)}$. In this case the roots of $x^2+x+1=0$ being impossible, α, β, suppose, we can only separate x^2+x+1 into impossible factors $(x-\alpha)(x-\beta)$: we might do this and assume as before, but the following method is preferable in such cases.

Assume $\dfrac{3x^2+5x+1}{x(x^2+x+1)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+x+1}=\dfrac{1}{x}+\dfrac{Bx+C}{x^2+x+1}$, by Ex. 5, (where the latter fraction may be considered as arising from the addition of the two with impossible den^{rs}:) now to obtain B and C, multiply by x^2+x+1, and then put $x^2+x+1=0$, or $x^2=-x-1$;

then we have $\dfrac{3(-x-1)+5x+1}{x}$ or $\dfrac{2x-2}{x}=Bx+C$;

$\therefore\ 2x-2=B(-x-1)+Cx$, and, (equating coeff^s) $B=2$, $C=4$.

Ex. 8. Let

$$\frac{2x^3+3x^2+4}{(x^2+1)(x^2-x+1)^3}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2-x+1)^3}+\frac{Ex+F}{(x^2-x+1)^2}+\frac{Gx+H}{x^2-x+1},$$

(the reason for which assumption is plain from Ex. 6;) then to find A and B, mult. by x^2+1, and put $x^2+1=0$, or $x^2=-1$;

$$\therefore\ \frac{-2x-3+4}{(-x)^3}\ \text{or}\ \frac{-2x+1}{x}=Ax+B,$$

whence $-2x+1=-A+Bx$, and $A=-1$, $B=-2$:

to find C and D, multiply by $(x^2-x+1)^3$, and put $x^2-x+1=0$, or $x^2=x-1$; then $\dfrac{2x(x-1)+3(x-1)+4}{x}$ or $\dfrac{3x-1}{x}=Cx+D$,

$\therefore\ 3x-1=C(x-1)+Dx$, and $C=1$, $D=2$; and now, before we can find E and F, we must subtract $\dfrac{Cx+D}{(x^2-x+1)^3}$ or $\dfrac{x+2}{(x^2-x+1)^3}$ from the first side, and then shall be able to strike out of both numerator and denominator the factor x^2-x+1.

Ex. 36.

Resolve into partial fractions,

1. $\dfrac{2x-3}{x(x^2+1)}$. 2. $\dfrac{x-2}{x^4-1}$. 3. $\dfrac{x^2+1}{x^3-1}$. 4. $\dfrac{2x^2-1}{x^3+1}$.

5. $\dfrac{1}{x(x^2-1)}$. 6. $\dfrac{3x-2}{x^2(x^2+1)^2}$. 7. $\dfrac{3x^3-x^2+3}{x(x^2-x+1)^2}$. 8. $\dfrac{x^4}{(x^4-1)^2}$.

Def. A series is said to be *convergent* or *divergent*, according as there exists or not a *limit* to the sum of its terms taken *ad infinitum:* thus $1+\frac{1}{2}+\frac{1}{4}+$ &c. for which $\Sigma=2$, is convergent; but $1+2+4+$&c. is divergent.

173. Any series is convergent, in which, after some finite No. of terms, the ratio of each term to the preceding becomes <1.

For let $S+a+b+c+$ &c. represent the series, S denoting the sum of that part of the series before the point in question, and which will of course be finite since the No. of its terms is finite; and let each of the ratios $\frac{b}{a}$, $\frac{c}{b}$, &c. be <1: suppose k to be $>$ the greatest of them; then $b<ka$, $c<kb<k^2a$, &c.,

$$\text{and } \Sigma < S + a(1+k+k^2+\text{\&c.}) < S + \frac{a}{1-k},$$

which is finite since $k<1$.

Ex. The series for e [117], viz. $1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+$ &c. is convergent, since the ratios in question are 1, $\frac{1}{2}$, $\frac{1}{3}$, &c.

174. The sum *ad inf.* of any series, such as $a-b+c-d+e-$&c., where the terms a, b, c, &c. continually decrease, as before, and are alternately positive and negative, lies between a and $a-b$.

For $\Sigma=(a-b)+(c-d)+$&c., in which the terms are all *positive*, and is, therefore, $>a-b$; but $\Sigma=a-(b-c)-(d-e)-$&c., in which the terms are all *negative*, except the first, and therefore is $<a$.

Ex. $\Sigma=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+$ &c. lies between 1 and $\frac{1}{2}$.

175. If R, S, be the greatest and least ratios formed by dividing each coeff. of the series $a+bx+cx^2+$ &c. by the preceding coeff., then, as in [173], the sum of the series lies between

$$a(1+Rx+R^2x^2+\text{\&c.}) \text{ and } a(1+Sx+S^2x^2+\text{\&c.}),$$

or, when Rx (and therefore Sx) is <1, between $\frac{a}{1-Rx}$ and $\frac{a}{1-Sx}$.

Hence if in any case $Rx<1$, the series will be convergent; if $Rx>1$ but $Sx<1$, the series may or may not be convergent but if $Sx>1$, it must be divergent. And it is plain that this statement is also true when R, S, are the greatest and least ratios, formed as above, after any finite No. of terms of the series.

Ex. 1. In the series for $(1+x)^n$, the inverse ratio of two consecutive coeff^s is $\frac{n-r+1}{r}=\frac{n+1}{r}-1$: hence $R=(n+1)-1=n$,

and $S = -1$; so that, when $nx < 1$ or $x < 1 \div n$, the value of $(1+x)^n$ will lie between $\dfrac{1}{1-nx}$ and $\dfrac{1}{1+x}$.

Ex. 2. What error will be made in taking the first three terms of the series $2(1+\frac{1}{4})^{\frac{1}{5}}$ as the value of $\sqrt[5]{40}$?

The *fourth* term is .0015; hence the sum of the series after the third term lies between $\dfrac{.0015}{1-\frac{1}{5}\times\frac{1}{4}}$ and $\dfrac{.0015}{1+\frac{1}{4}}$, that is, between .0015 &c. and .0012, which are therefore the limits of the error.

176. A *recurring* series is one in which any term is the sum of one or more preceding ones with constant coefficients, or *in which an equation of the first degree with constant coefficients exists between any given number of consecutive terms.*

Ex. 1. In the Geom. series $1 + 2x + 4x^2 +$ &c., each term is formed from the preceding by means of the constant factor $2x$, so that if u_r, u_{r+1} denote the r^{th} and $(r+1)^{\text{th}}$ terms, $u_{r+1} = 2xu_r$, or $u_{r+1} - 2xu_r = 0$, in which equation the sum of the coeff$^{\text{s}}$ or $1-2x$ is called the *Scale of Relation* of the series.

Ex. 2. In $3 + 11x + 31x^2 + 95x^3 + 283x^4 +$ &c., it will be found that any *three* consecutive terms are connected by the equation

$$u_{r+2} = 2xu_{r+1} + 3x^2u_r \text{ or } u_{r+2} - 2xu_{r+1} - 3x^2u_r = 0:$$

thus $31x^2 = 2x.11x + 3x^2.3$, $95x^3 = 2x.31x^2 + 3x^2.11x$, &c.: and here the scale of relation is $1 - 2x - 3x^2$.

177. In any given case of a recurring series it will be generally easy to find by trial its scale of relation, as it will probably consist of very few terms. Thus in [176 Ex. 2], since we see at once that the scale cannot consist of two terms only (for then, as in Ex. 1, the series would be in G. P.) we may assume it to be of the form $1 + ax + bx^2$; then we shall have the equations

(i) $31+11a+3b=0$, (ii) $95+31a+11b=0$, (iii) $283+95a+31b=0$;

and since the values $a=-2$, $b=-3$, obtained from (i) and (ii), also satisfy (iii), we conclude that the scale of relation *is* of the assumed form, viz. (as above) $1-2x-3x^2$.

Of course, if we had found that the values of a and b obtained from (i) and (ii) did not also satisfy (iii), it would have shewn that our first assumption was wrong, and we should have had to begin again by assuming the scale to be of the form $1+ax+bx^2+cx^3$.

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178. Series of the above kind will always result from the expansion of an algebraical proper fraction, the denominator itself being the scale of relation.

Thus we have seen [172 Ex. 1] that

$$\frac{a' + b'x}{1 + ax + bx^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.,$$

where A and B are given by the equations $A = a'$, $B + aA = b'$, and then, for the other coefficients,

$$C + aB + bA = 0,\ D + aC + bB = 0,\ E + aD + bC = 0,\ \&c.;$$

that is, $A + Bx + Cx^2 + \&c.$ will be a recurring series, whose scale of relation is $1 + ax + bx^2$.

179. Conversely, we may sum *ad infinitum* any such series, or restore it to the form of its generating fraction.

For let $A + Bx + Cx^2 + \&c.$ be a recurring series, and let its scale of relation, found as in [178], be $1 + ax + bx^2$: assume $\dfrac{a' + b'x}{1 + ax + bx^2} = A + Bx + Cx^2 + \&c.$; then [172] $a' = A$, $b' = B + aA$; and $\therefore A + Bx + \&c.$ *ad inf.* $= \Sigma = \dfrac{A + (B + aA)x}{1 + ax + bx^2} = \dfrac{A(1 + ax) + Bx}{1 + ax + bx^2}$.

Similarly, if the scale of relation had been $1 + ax + bx^2 + cx^3$, we should have found $\Sigma = \dfrac{A(1 + ax + bx^2) + Bx(1 + ax) + Cx^2}{1 + ax + bx^2}$.

Ex. In [177 Ex. 1], $\Sigma = \dfrac{A}{1 + ax} = \dfrac{1}{1 - 2x}$;

in Ex. 2, $\Sigma = \dfrac{A(1 + ax) + Bx}{1 + ax + bx^2} = \dfrac{3(1 - 2x) + 11x}{1 - 2x - 3x^2} = \dfrac{3 + 5x}{1 - 2x - 3x^2}$.

180. *To find, generally, the Sum of a recurring series.*

Let the scale of relation be $1 + p_1x + p_2x^2 + \&c.\ p_nx^n$, and the series $u_0 + u_1x + u_2x^2 + \&c.$; and let S_1, S_2, &c. denote the sum of *one*, *two*, &c. terms, Σ the sum *ad infinitum*.

$$\begin{aligned} \text{Then we have}\quad & u_n + p_1xu_{n-1} + p_2x^2u_{n-2} + \&c. + p_nx^nu_0 = 0, \\ & u_{n+1} + p_1xu_n + p_2x^2u_{n-1} + \&c. + p_nx^nu_1 = 0, \\ & \&c. \qquad\qquad\qquad\qquad\qquad = \&c.; \end{aligned}$$

$\therefore$, adding, $(\Sigma - S_n) + p_1x(\Sigma - S_{n-1}) + p_2x^2(\Sigma - S_{n-2}) + \&c. + p_nx^n\Sigma = 0$,

$$\text{or } \Sigma = \frac{S_n + p_1xS_{n-1} + p_2x^2S_{n-2} + \&c.}{1 + p_1x + p_2x^2 + \&c.}$$

181. The general term of a recurring series, and its sum to n terms, may be easily found, whenever the scale of relation can be separated into simple factors.

Thus, if $1 + ax + bx^2 + cx^3 = (1 + \alpha x)(1 + \beta x)(1 + \gamma x)$,

then the fraction $\dfrac{a' + b'x + c'x^2}{1 + ax + bx^2 + cx^3}$ (which may represent the sum *ad inf.* of the series for which $1+ax+bx^2+cx^3$ is the scale of relation) may be separated into the partial fractions $\dfrac{A}{1+\alpha x} + \dfrac{B}{1+\beta x} + \dfrac{C}{1+\gamma x}$; and these, when expanded, will produce three *geometrical* series, the sum of which to n terms will give that of the recurring series, and the sum of their n^{th} terms its n^{th} term.

Ex. In [177 Ex. 2], we have

$$\Sigma = \frac{3 + 5x}{1 - 2x - 3x^2} = \frac{3 + 5x}{(1 - 3x)(1 + x)} = \frac{7}{2(1 - 3x)} - \frac{1}{2(1 + x)}$$

$$= \tfrac{7}{2}\{1 + 3x + (3x)^2 + \&c.\} - \tfrac{1}{2}\{1 - x + x^2 - \&c.\},$$

the n^{th} term of which is $\frac{1}{2}\{7.3^{n-1} - (-1)^{n-1}\}x^{n-1}$,

and the sum of n terms $\dfrac{7}{2} \cdot \dfrac{(3x)^n - 1}{3x - 1} + \dfrac{1}{2} \cdot \dfrac{(-x)^n - 1}{x + 1}$.

The above method may of course be employed whenever the den^r admits of separation into simple factors: but the result would appear in a complicated form if the factors be not rational.

182. In other cases, if we can succeed in finding by any means the n^{th}, or *general*, term of the series, we may, by using this to obtain the $(n + 1)^{\text{th}}$, &c. terms, obtain the sum to n terms as follows.

Let S denote the sum to n terms of the recurring series,

$$A + Bx + Cx^2 + \&c. + Nx^{n-1} + Px^n + Qx^{n+1} + \&c.,$$

whose scale of relation is $1 + ax + bx^2$, and Σ' the sum *ad inf.* of the same series *after* the first n terms: then since the latter portion of the given series, viz. $Px^n + \&c.$, is also a recurring series, with the *same* scale of relation $1 + ax + bx^2$, we have

$$S = \Sigma - \Sigma' = \frac{\{A(1 + ax) + Bx\} - \{P(1 + ax) + Qx\}x^n}{1 + ax + bx^2}.$$

Ex. In $2 - x + x^3 - 2x^4 + \&c.$, we shall find the scale of relation to be $1 + 2x + x^2$, and therefore

$$\Sigma = \frac{2(1+2x) - x}{1 + 2x + x^2} = (2 + 3x)(1 + x)^{-2}$$
$$= (2 + 3x)(1 - 2x + 3x^2 - \&c),$$

and $\therefore P =$ coeff. of $x^n = (-1)^n \{2(n+1) - 3n\} = (-1)^n (2 - n)$, and Q (putting $n + 1$ for n) $= (-1)^{n+1}(1 - n) = (-1)^n (n - 1)$:

hence $$S = \frac{\{2(1+2x) - x\} + (-1)^n \{(n-2)(1+2x) - (n-1)x\} x^n}{1 + 2x + x^2}$$
$$= \frac{(2 + 3x) + (-1)^n \{n - 2 + (n-3)x\} x^n}{1 + 2x + x^2}.$$

183. If a mere numerical series be given, as $1+5+13+29+61+\&c.$, then we may first sum the series $1 + 5x + 13x^2 + \&c.$, the n^{th} term of which we shall find (as above) to be $(4.2^{n-1} - 3)x^{n-1}$, and the sum to n terms $4\,\dfrac{(2x)^n - 1}{2x - 1} - 3\,\dfrac{x^n - 1}{x - 1}$; whence putting $x = 1$, (and first striking out from num^r^ and den^r^ of the latter fraction the vanishing factor $x - 1$) we obtain for the n^{th} term of the given series $4.2^{n-1} - 3$, and for its sum to n terms, $4(2^n - 1) - 3n$.

In this case, however, it might have been noticed that the 1^st^ term of the given series $= 4 - 3$, the 2^nd^ $= 4.2 - 3$, the 3^rd^ $= 4.4 - 3$, and so the general term $= 4.2^{n-1} - 3$, and the sum of n terms $= 4(1 + 2 + 4 + \&c. + 2^{n-1}) - 3n = 4(2^n - 1) - 3n$. A similar artifice may be used to sum other series of this kind.

Ex. 37.

Sum *ad infinitum* the recurring series

1. $1 + 2x + 3x^2 + 5x^3 + 8x^4 + \&c.$
2. $1 + 6x + 9x^2 - 27x^4 - \&c.$
3. $1 + 3x + 7x^2 + 13x^3 + 25x^4 + 51x^5 + 103x^6 + \&c.$
4. $1 + 4x + 6x^2 + 11x^3 + 28x^4 + 63x^5 + 131x^6 + \&c.$

Sum *ad infinitum*, and find the general term of the series

5. $1 - 3x + 5x^2 - 7x^3 + 9x^4 - \&c.$
6. $1 + 5x + 15x^2 + 31x^3 + 53x^4 + 81x^5 + 115x^6 + \&c.$

Sum to n terms the series

7. $1 + 6x + 12x^2 + 48x^3 + 120x^4 + \&c.$
8. $2 - 5x + 9x^2 - 19x^3 + 37x^4 - \&c.$
9. $1 + 3 + 7 + 15 + 31 + \&c.$
10. $1 + 2 + 5 + 14 + 41 + \&c.$
11. $1 + 3 + 5 + 11 + 21 + \&c.$
12. $1 + 2 + 2\frac{1}{3} + 2\frac{4}{9} + 2\frac{13}{27} + \&c.$

184. Certain series may be summed by the following method.

Ex. 1. Let $S = a + (a+b)x + (a+2b)x^2 + \&c. + \{a+(n-1)b\}x^{n-1}$;

then $Sx = \qquad ax + (a+b)x^2 + \&c. + \{a+(n-2)b\}x^{n-1} + \{a+(n-1)b\}x^n$;

and $S(1-x) = a + bx + bx^2 + \&c. + bx^{n-1} + bx^n - (a+nb)x^n$,

$$= a(1-x^n) + bx \cdot \frac{1-x^n}{1-x} - nbx^n;$$

$$\therefore S = \left\{\frac{a}{1-x} + \frac{bx}{(1-x)^2}\right\}(1-x^n) - \frac{nbx^n}{1-x},$$

and Σ (found by putting $n = \infty$, or $x^n = 0$) $= \dfrac{a}{1-x} + \dfrac{bx}{(1-x)^2}$.

In the above result are included all series, in which the terms consist of two factors, proceeding one in A.P., the other in G.P.

Ex. 2. Sum $\dfrac{1}{a(a+b)} + \dfrac{1}{(a+b)(a+2b)} + \&c.$

Let $S' = \dfrac{1}{a} + \dfrac{1}{a+b} + \&c. + \dfrac{1}{a+(n-1)b} + \dfrac{1}{a+nb}$;

$\therefore S' - \dfrac{1}{a} = \dfrac{1}{a+b} + \dfrac{1}{a+2b} + \&c. + \dfrac{1}{a+nb}$;

hence $\dfrac{1}{a} = \dfrac{b}{a(a+b)} + \dfrac{b}{(a+b)(a+2b)} + \&c. + \dfrac{b}{\{a+(n-1)b\}\{a+nb\}} + \dfrac{1}{a+nb}$

$= bS + \dfrac{1}{a+nb}$; $\therefore S = \dfrac{n}{a(a+nb)}$, $\Sigma = \dfrac{1}{a(an^{-1}+b)} = \dfrac{1}{ab}$, when $n = \infty$.

So too we may sum $\dfrac{1}{a(a+b)(a+2b)} + \dfrac{1}{(a+b)(a+2b)(a+3b)} + \&c.$,

and, generally, any series of such fractions, in which it will be observed that the den^rs are formed of factors in A.P., by assuming S' to represent $n+1$ terms of a similar series, wanting the last factors in the den^rs. This method also (called the method of Subtraction) will be found to succeed in some other instances.

Ex. 3. To sum the series $1^2 + 2^2 + 3^2 + \&c.$ to n terms.

Assume $S = 1^2 + 2^2 + \&c. + n^2 = A + Bn + Cn^2 + Dn^3 + \&c.$:

then, writing $n+1$ for n, we obtain

$$1^2 + 2^2 + \&c. + n^2 + (n+1)^2 = A + B(n+1) + C(n+1)^2 + D(n+1)^3 + \&c.;$$

$\therefore$ subtracting, $\qquad (n+1)^2 = B + C(2n+1) + D(3n^2 + 3n + 1)$,

[where we have stopped with D, because there are no terms on

the first side with n^3, &c., and therefore the coeff^s E, F, &c. on the second side would have to be equated to zero:]
hence, equating coefficients, $1 = 3D$, $2 = 2C + 3D$, $1 = B + C + D$, whence we get $D = \frac{1}{3}$, $C = \frac{1}{2}$, $B = \frac{1}{6}$, and $S = A + \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$: put $n = 1$; then S or $1^2 = A + \frac{1}{6} + \frac{1}{2} + \frac{1}{3}$, whence we find that $A = 0$, and, therefore, $S = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 = \frac{1}{6}n(n+1)(2n+1)$.

Since when $n = 0$, the sum of the series ought to reduce itself to *zero*, we might have seen at once that there would have been no constant term, and might have assumed, therefore,

$$S = An + Bn^2 + Cn^3 + \&c.$$

Ex. 4. Similarly, it may be shewn that $1^3 + 2^3 + 3^3 + \&c.$ to n terms $= \{\frac{1}{2}n(n+1)\}^2 = \{1 + 2 + 3 + \&c. \text{ to } n \text{ terms}\}^2$.

185. We may sum many series by the method of *Increments*.

Def. The *increment* of any quantity, depending upon n, is that quantity by which it is increased (or diminished) when for n we write $n+1$. Denoting this increment by D, and the two values of the given quantity, which is called the *integral*, by S_n, S_{n+1}, we have $S_{n+1} - S_n = D$.

186. Now (i) if $D = A$, a constant, then the integral to which it belongs, or S_n, $= An + C$, where C may be any constant whatever.

For $S_{n+1} = A(n+1) + C$; $\therefore$ $S_{n+1} - S_n = An = D$.

187. (ii) If D can be expressed in the form $A.u_1.u_2 \ldots u_{m-1}.u_m$, where u_1, u_2, &c. u_m, represent m factors in A.P., and such that, when n is changed to $n + 1$, they become each changed to the consecutive factor (as, for instance, when $D = n(n+1)$, or $= (n-1)\,n\,(n+1)$, or $= (2n+1)(2n+3)$, &c.,)

$$\text{then } S_n = \frac{A}{(m+1)\,b}\,u_0.u_1.u_2 \ldots u_{m-1}.u_m + C,$$

where u_0 represents the factor immediately preceding u_1, and b the common difference of the factors.

For thus we should have

$$S_{n+1} = \frac{A}{(m+1)\,b}\,u_1.u_2.u_3 \ldots u_m.u_{m+1} + C;$$

$$\therefore\; S_{n+1} - S_n = \frac{A}{(m+1)\,b}\,u_1.u_2.u_3 \ldots u_m\,(u_{m+1} - u_0) = A.u_1.u_2.u_3 \ldots u_m = D.$$

188. The value of the constant C may always be determined in any given case, as follows, where S_n will represent the sum of n terms, and therefore D, the $(n+1)^{\text{th}}$ term.

Ex. 1. $1+2+3+\&c.$; here $D=n+1$, and $S_n=\frac{1}{2}n(n+1)+C$: to find C, put $n=1$; then S_1, or $1, =1+C$, and $C=0$; $\therefore S_n=\frac{1}{2}n(n+1)$.

Ex. 2. $1^2+2^2+3^2+\&c.$; here $D=(n+1)^2=n(n+1)+n+1$;
$\therefore S_n=\frac{1}{3}(n-1)\,n\,(n+1)+\frac{1}{2}n(n+1)$ ($C=0$, as in Ex. 1.)
$=\frac{1}{6}n(n+1)(2n+1)$, as in [184 Ex. 3].

To save trouble, we shall express the value of C, thus found, in a bracket, as $[\frac{1}{3}]$; unless it be zero, when we shall omit to notice it.

Ex. 3. Find the sum of n terms of $1.3+3.5+5.7+\&c.$

Here $D=(n+1)^{\text{th}}$ term $=(2n+1)(2n+3)$;

$$\therefore S=\frac{(2n-1)(2n+1)(2n+3)}{3.2}+[\tfrac{1}{2}]=\tfrac{1}{3}n(4n^2+6n-1),$$

where the factor 2 in the den^r arises from the common diff. b, which in this case is 2.

Ex. 4. Find the sum of n terms of any order of Figurate Nos.

Figurate Nos. are formed as follows, the n^{th} term of each order being the sum of n terms of the preceding order.

1st order	1, 1, 1, 1, &c.	Now for the 1st order, $S_n=n$, and
2nd	1, 2, 3, 4, &c.	this is the n^{th} term of the 2nd order;
3rd	1, 3, 6, 10, &c.	$\therefore D$ for 2nd $=n+1$, and $S_n=\frac{1}{2}n(n+1)$;

hence D for 3rd order $=\dfrac{(n+1)(n+2)}{1.2}$, and $S_n=\dfrac{n(n+1)(n+2)}{1.2.3}$;

and generally for r^{th} order, $S_n=\dfrac{n(n+1)\dots(n+r-1)}{1.2\dots r}$.

Ex. 5. To find the sum of n terms of any order of Polygonal Nos.

If in the expression $S=n+\frac{1}{2}n(n-1)b$, which is the sum of n terms of an AR. series whose first term is *unity*, we give b the values 0, 1, 2, &c. successively, we get the general term of the 1st, 2nd, 3rd, &c. orders of *Polygonal* Nos., and from these the Nos. of each order are found by giving n the values 1, 2, 3, &c. Thus we have

1st order, for which the n^{th} term	$=n$	1, 2, 3, 4, &c.
2nd order, ...	$=\frac{1}{2}n(n+1)$,	1, 3, 6, 10, &c.
3rd order, ...	$=n^2$,	1, 4, 9, 16, &c.
4th order, ...	$=\frac{1}{2}n(3n-1)$,	1, 5, 12, 22, &c.

The Nos. in each row have received respectively the names *linear*, *triangular*, *square*, *pentagonal*, &c. and generally of *polygonal* Nos. from the fact that (putting a dot to represent each unit) the Nos. of

each kind admit of being arranged in the form of the corresponding polygon: thus (i) *linear*, ... &c.; (ii) *triangular*, . ∴ ∴̇. &c.; (iii) *square*, . ∷ ⁝⁝⁝ &c.

Now the general term of the r^{th} order will be $n+\frac{1}{2}n(n-1)(r-1)$, and, therefore, $D=(n+1)+\frac{1}{2}(r-1)n(n+1)$, and $S_n=\frac{1}{2}n(n+1)$ $+\frac{1}{6}(r-1)(n-1)n(n+1)=\frac{1}{6}n(n+1)\{(r-1)(n-1)+3\}$.

Hence for linear numbers $S_n=\frac{1}{2}n(n+1)$, for triangular $S_n=\frac{1}{6}n(n-1)(n+2)$, for square $S_n=\frac{1}{6}n(n+1)(2n+1)$ &c.

N.B. The above results may be applied to calculate the No. of cannon balls piled in a pyramidal heap. Thus if the base be triangular and one side of it contain 30 balls, the No. of shot in the pile will be $\frac{1}{6}n(n+1)(n+2)$ when $n=30$, that is, 4960.

If the pile be not pyramidal, but formed of horizontal rectangular courses, finishing by a single row at the top, then, if l be the length of the top row, the length of the $(n+1)^{\text{th}}$ course from the top will be $l+n$, and the breadth $n+1$;

$$\text{hence } D=(l+n)(n+1)=n(n+1)+l(n+1):$$

$$\therefore\ S_n=\tfrac{1}{3}n(n^2-1)+\tfrac{1}{2}ln(n+1)=\tfrac{1}{6}n(n+1)(3l'-n+1),$$

where $l'=l+n-1=$ length of lowest row, and $n=$ the breadth.

If we suppose the pile, in any case, a broken one, the first m courses being removed, then the No. of balls $=S_n-S_m$.

Ex. 6. Find the sum of n terms of $5+9+16+26+39+$ &c.

If we set down this series and take the *differences* of each two consecutive terms, and then the differences of these, and so on, we shall at last arrive at a row of differences all identical.

Thus 5, 9, 16, 26, 39, &c.
4, 7, 10, 13, &c.
3, 3, 3, &c.

Now in any row of these differences, the n^{th} term is the increment of the n^{th} term of the preceding row; hence $3=D_1$, and the integral of this $=3n+[1]=(3n+1)=D_2$; the int. of this $=\frac{1}{6}(3n-2)(3n+1)+[\frac{13}{3}]=\frac{1}{2}(3n^2-n+8)=D_3$, $=$ the n^{th} term of the given series; hence the $(n+1)^{\text{th}}$ term, or D,

$$=\tfrac{1}{2}(3n^2+5n+10)=\tfrac{3}{2}(n^2+\tfrac{5}{3}n+\tfrac{10}{3})=\tfrac{3}{2}\{n(n+1)+\tfrac{2}{3}(n+5)\};$$

$$\therefore\ S_n=\tfrac{1}{2}(n-1)n(n+1)+\tfrac{1}{2}(n+4)(n+5)-[10]=\tfrac{1}{2}n(n^2+n+8).$$

It will generally be easy to throw the value of D into the form of factors, as above; if not, we can always do it by Ind. Coefficients:

thus, if $D = n^2 + 1$, put $n^2 + 1 = An(n+1) + Bn + C$, and equate coeff^s. But here we might have written $n^2+1 = n(n-1)+(n+1)$, which is at once of the proper form.

N.B. The above method may be applied, whenever any order of differences becomes at length composed of the *same* quantities.

189. Generally, let a, b, c, d, &c. be any series, and let d_1, d_2, &c. be the first terms of the 1st, 2nd, &c. orders of differences, found as above: suppose that in the 3rd order the differences become identical, and each $= d_3$; then we shall have $D_1 = d_3$, $D_2 = nd_3 + C$: but (putting $n=1$) $d_2 = d_3 + C$; $\therefore D_2 = (n-1)d_3 + d_2$: hence $D_3 = \frac{1}{2}(n-1)(n-2)d_3 + nd_2 + C$, and (as before) $d_1 = d_2 + C$;

$\therefore D_3 = \frac{1}{2}(n-1)(n-2)d_3 + (n-1)d_2 + d_1$; and, in like manner,

$$D_4 = \frac{(n-1)(n-2)(n-3)}{1.2.3}d_3 + \frac{(n-1)(n-2)}{1.2}d_2 + (n-1)d_1 + a$$

$= n^{\text{th}}$ term of given series:

$$\therefore D = (n+1)^{\text{th}} \text{ term} = a + nd_1 + \frac{n(n-1)}{1.2}d_2 + \frac{n(n-1)(n-2)}{1.2.3}d_3,$$

$$\text{and } S = na + \frac{n(n-1)}{1.2}d_1 + \frac{n(n-1)(n-2)}{1.2.3}d_2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}d_3.$$

A similar formula for S will evidently hold, whatever be the number of orders of differences, that is, of the quantities d_1, d_2, &c.

Ex. In [188 Ex. 6]

$$S = 5n + 2n(n-1) + \tfrac{1}{3}n(n-1)(n-2) = \tfrac{1}{3}n(n^2+n+8).$$

190. (iii) If $D = \dfrac{A}{u_1.u_2\ldots u_m}$, then $S_n = -\dfrac{A}{(m-1)b.u_1.u_2\ldots u_{m-1}} + C.$

$$\text{For } S_{n+1} = -\frac{A}{(m-1)b.u_2.u_3\ldots u_m} + C; \therefore S_{n+1} - S_n = \frac{A(u_m - u_1)}{(m-1)bu_1.u_2\ldots u_m} = D.$$

Ex. 1. $\dfrac{1}{1.3} + \dfrac{1}{3.5} + \dfrac{1}{5.7} + \text{\&c.}$: $D = \dfrac{1}{(2n+1)(2n+3)}$;

$$\therefore S_n = -\frac{1}{2(2n+1)} + [\tfrac{1}{2}] = \frac{n}{2n+1},\ \Sigma = \frac{1}{2+n^{-1}} = \frac{1}{2}, \text{ when } n = \infty.$$

Ex. 2. $\dfrac{1}{1.3.5} + \dfrac{2}{3.5.7} + \dfrac{3}{5.7.9} + \text{\&c.}$: $D = \dfrac{n+1}{(2n+1)(2n+3)(2n+5)}$

$$= \frac{\frac{1}{2}(2n+5-3)}{(2n+1)(2n+3)(2n+5)} = \frac{1}{2(2n+1)(2n+3)} - \frac{3}{2(2n+1)(2n+3)(2n+5)};$$

$$\therefore S_n = -\frac{1}{4(2n+1)} + \frac{3}{8(2n+1)(2n+3)} + [\tfrac{1}{8}], \text{ and } \Sigma = \tfrac{1}{8}.$$

Ex. 38.

Sum to n terms, and, when possible, $(x < 1)$ *ad infinitum*.

1. $1 + 2x + 3x^2 + 4x^3$ &c.
2. $1 + 4x + 7x^2 + 10x^3 +$ &c.
3. $1 + 3.2 + 5.2^2 + 7.2^3 +$ &c.
4. $1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} +$ &c.
5. $a + 2(a + x) + 4(a + 2x) +$ &c.
6. $a - (a + 1)x + (a + 2)x^2 -$ &c.
7. $1.2 + 2.3 + 3.4 +$ &c.
8. $1.3 + 2.4 + 3.5 +$ &c.
9. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} +$ &c.
10. $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} +$ &c.
11. $\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} +$ &c.
12. $\frac{3}{1.2.3} + \frac{5}{2.3.4} + \frac{7}{3.4.5} +$ &c.
13. $1^2 + 3^2 + 5^2 +$ &c.
14. $1^4 + 2^4 + 3^4 +$ &c.
15. $1^3 + 3^3 + 5^3 +$ &c.
16. $1^2.2 + 2^2.3 + 3^2.4 +$ &c.
17. $\frac{1}{2.4} + \frac{1}{4.6} + \frac{1}{6.8} +$ &c.
18. $\frac{1}{8.18} + \frac{1}{10.21} + \frac{1}{12.24} +$ &c.
19. $\frac{1}{1.3.4} + \frac{1}{2.4.5} + \frac{1}{3.5.6} +$ &c.
20. $\frac{1.4}{2.3} + \frac{2.5}{3.4} + \frac{3.6}{4.5} +$ &c.
21. Find the No. of balls (i) in a triangular, (ii) in a square pile, each side of the base containing 25 balls.
22. Find the No. in each of the above cases, if the piles be incomplete, 10 courses having been removed.
23. Find the No. in a rectangular pile, the length and breadth of the base row being 50 and 24.
24. Find the No. in (23), if the pile be incomplete, half the whole number of courses having been removed.

CHAPTER XI.

PROPERTIES OF NUMBERS, AND DIOPHANTINE PROBLEMS.

191. If n be any No., then any other No. N may be expressed in the form $N = np + r$, where p is the integral-quotient upon dividing N by n, and r the rem^r will be less than n. The No. n in such a case, to which the other is referred, is called the *Modulus;* and, to any given Mod. n, there are evidently (including the case of $r = 0$) n different *Forms* of numbers, with different values of r.

Thus, to Mod. 2, any No. whatever will be expressed in one of the *two* forms, $2p$, $2p + 1$, according as the No. is even or odd; to Mod. 3, in one of the *three* forms, $3p$, $3p + 1$, $3p + 2$, or $3p$, $3p \pm 1$, (since $3p + 2$ may be expressed as $3(p + 1) - 1$ or $3p' - 1$); so also to Mod. 4, in one of the *four* forms, $4p$, $4p \pm 1$, $4p + 2$; and generally, to Mod. n in one of the n forms, np, $np \pm 1$, $np \pm 2$, &c., as far as $np + \frac{1}{2}n$ or $np \pm \frac{1}{2}(n - 1)$, according as n is even or odd.

We may apply the above to the following Examples.

Ex. 1. Shew that the product of two even Nos. is even, and of two odd Nos. odd; and hence that $N^n \pm N$ is *always* even.

(i) $2p \times 2p' = 4pp'$, which is even: (ii) $(2p + 1)(2p' + 1) = 4pp' + 2(p + p') + 1$, which is odd: hence, by induction, the product of any No. of even Nos. is even, and of odd Nos. odd: hence (iii) N^n is even or odd according as N is, and $\therefore N^n \pm N$ is always even.

Ex 2. If N be odd, shew that $(N^2 + 3)(N^2 + 7)$ is divisible by 32.

For let N (being odd) $= 2p + 1$; then $(N^2 + 3)(N^2 + 7) = (4p^2 + 4p + 4)(4p^2 + 4p + 8) = 16(p^2 + p + 1)(p^2 + p + 2)$: but (Ex. 1) $p^2 + p$, and, therefore, $p^2 + p + 2$ is *even:* hence the given quantity is div. by 32.

Ex. 39.

1. The sum or difference of any two odd Nos. is even; and the sum of two consecutive odd Nos. is divisible by 4.

2. The product of two consecutive Nos. is even; and the product of two consecutive even Nos. is divisible by 8.

3. The product of any three consecutive Nos. is div. by 1.2.3, and of any four, by 1.2.3.4.

4. If an odd No. divides an even No., it will also divide its half.

5. The difference of the squares of any two odd Nos. is div. by 8, and the sum of the squares of three consecutive odd Nos. is of the form $12p - 1$.

6. If n be even $n(n^2 - 4)$ is divisible by 48; and if n be odd, $n(n^2 - 1)$ is divisible by 24.

7. Every even square is of the form $16p$ or $4(4p+1)$, and every odd square of the form $8p + 1$.

8. Every No. and its cube, when div. by 6, leave the same rem[r].

9. Shew from (7) that, if the sum of an odd and even square be a square, the even square is divisible by 16.

10. The product of any two odd Nos. is less than the square of the middle No. between them by the square of half their difference.

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192. A *prime* number is one that admits of no divisors but unity; all others, being composed of factors, are called *composite* Nos.

The following are elementary propositions on prime Nos.

(i) *If* a *be prime to* b, *then* a ± b *is prime to each of them.*

For, if not, let $a \pm b$ and a have a common factor, then (63) $(a \pm b) \sim a$, or b, has the same common factor, which is absurd, since a is prime to b.

(ii) *If* a *be prime to* b, *then* a + b *is prime to* a − b, *or else they have only the common factor* 2.

For, whatever common factor $a+b$ and $a-b$ may have, the same (63) will their sum or difference, $2a$ and $2b$, have; and these, since a is prime to b, can have only the common factor 2.

(iii) *If* a *be prime to* b, *and* a = nb + r, *then* r *is also prime to* b.

For if b and r had any common factor, then (63) $nb + r$, and $\therefore$ a, would have the same.

(iv) *If* a *be prime to each of* b *and* c, *then* a *is prime to* bc.

For let c be the *least* factor, prime to a, which will make bc div. by a, and let $c = na + r$, where $r < a$, and (iii) is also prime to a: then $bc = bna + br$; and, since bc and bna have a common factor a, $\therefore$ by (63) br has the same, that is, there is a factor (r), *less* than c which makes br divisible by a, which is absurd.

Conversely, if a be prime to bc, it is prime to each of b and c.

(v) Hence, if a be prime to b, c, d, &c. it is prime to bcd, &c.: and, conversely, if a be prime to any number, it is prime to each of its factors.

(vi) Hence, if a be prime to b, we have also $a \times a$ or a^2 prime to b, and $\therefore$ also a^3 prime to b, and, generally, a^m prime to b; and, since a^m is prime to b, therefore a^m is prime to $b \times b$ or b^2, and $\therefore$ also to b^3, and generally to b^n.

(vii) Since a fraction can only be reduced to lower terms, by dividing both num^r and den^r by a common factor, it follows that, if a be prime to b, $\frac{a}{b}$, and, generally, $\frac{a^m}{b^n}$, is in its lowest terms: hence also, if $\frac{a}{b}$ (a fraction in lowest terms) = any other fraction $\frac{c}{d}$, *then* c *and* d *must be equimultiples of* a *and* b.

(viii) Hence also, if a be an integer and $\sqrt[n]{a}$ a proper surd, it cannot be expressed as a fraction $\frac{p}{q}$: for then we should have $a = \frac{p^n}{q^n}$, or an integer = a fraction, which is absurd.

Now every proper surd (112) may be expressed with its num^r in the above form; and therefore no such surd can be expressed either as a fraction, or as a *terminating* or *circulating* decimal, either of which might be reduced to the form of a fraction.

193. If in either of the expressions $x^2 + x + 17$, $2x^2 + 29$, $x^2 + x + 41$, we give x successively the values 0, 1, 2, &c., we shall find that 17 terms of the first, 29 of the second, and 40 of the third, are prime numbers: the same is true for 31 terms of $2^x + 1$, if we give to x the values 1, 2, 2^2, &c.: but after these points the expressions no longer give primes, and it is easy to shew that *no such formula can give prime numbers only.*

For let $P = a + bx + cx^2 + \text{\&c.} = a + x(b + cx + \text{\&c.}) = a + mx$ suppose, and let P be a prime for a certain value x' of x: for x write $x' + nP$; then we have the new value of P or $P' = a + m(x' + nP) = P(1 + mn)$, which is no longer a prime.

194. *The No. of primes is infinite.*

For let P_1, P_2, &c. P_n be prime Nos. in order of magnitude: then the product $P_1.P_2...P_n$ is divisible by each of them, and $\therefore$ $P_1.P_2...P_n + 1$ by none of them: hence this No. must be either itself a prime, or divisible by some prime $> P_n$; and in either case it follows that P_n cannot be the greatest prime No.

195. The student should here refer to the proof in [71], that if a be a prime to b, then each of the quantities a, $2a$, $3a$, &c., $(b-1)a$, when divided by b, will leave a different positive rem^r.

196. *If* n *be a prime No., the coeff. of every term of* $(1+x)^n$, *except the first and last, is divisible by* n.

For [92 Cor. 2] $n(n-1)\ldots(n-r+1)$ is divisible by $1.2\ldots r$: therefore, if n be prime, and $r < n$, the product $(n-1)\ldots(n-r+1)$ must be div. by $1.2\ldots r$; and hence the coeff. of every term (except the first and last) will contain a factor n, and be divisible by it.

The same is obviously true of the coeff. of $(a+b+c)^n$, when n is a prime No.: since $\{(a+b)+c\}^n = (a+b)^n + C_1(a+b)^{n-1}c + \&c.$, where the coeff^s of $(a+b)^n$, as well as C_1, C_2, &c., come under the preceding case, and those of $(a+b)^{n-1}$, &c. are all integral; therefore the coeff. of every term of the expression, except a^n, b^n, c^n, is div. by n: and similarly for any such multinomial.

197. *Fermat's Theorem. If* n *be a prime No., and* N *prime to* n, *then* $N^{n-1}-1$ *is a multiple of* n.

For [196] $(a+b+c+\&c.)^n - (a^n+b^n+c^n+\&c.)$ is div. by n: suppose there are N quantities a, b, c, &c., and put each of them $=1$; then $N^n - N = N(N^{n-1}-1)$ is div. by n: but N is prime to n; $\therefore N^{n-1}-1$ is a multiple of n, or N^{n-1} is of the form $np+1$.

Upon the above result we may make the following remarks.

(i) N may be *any* No. $< n$, since each of these will be prime to n:

(ii) N^n and N, when divided by n, leave the same rem^r:

(iii) N^{n-1} is of the form $np+1$, if N be prime to n, and of the form np, if N be not prime to n, since in this case (n being prime) N must be a multiple of n; thus N^2 is of the form $3p$ or $3p+1$, according as N is a multiple of 3 or not:

(iv) Since n is prime, $n-1$ is even; and therefore $N^{n-1}-1 = (N^{\frac{1}{2}(n-1)}+1)(N^{\frac{1}{2}(n-1)}-1)$: hence, if N be prime to n, one of these two factors must be div. by n, and therefore $N^{\frac{1}{2}(n-1)}$ is of the form np or $np \pm 1$, according as N is a multiple of n or not; thus N^2 is of the form $5p$ or $5p \pm 1$, according as N is a multiple of 5 or not.

198. *Sir J. Wilson's Theorem. If* n *be a prime number, then* $1.2.3\ldots(n-1)+1$ *is a multiple of* n.

By [119] it is seen that

$1.2.3\ldots(n-1) = (n-1)^{n-1} - (n-1)(n-2)^{n-1} + \frac{1}{2}(n-1)(n-2)(n-3)^{n-1} - \&c.$;

but [197] each of $(n-1)^{n-1}$, $(n-2)^{n-1}$, &c., is of the form $np+1$;

hence $1.2.3\ldots(n-1) =$ (suppose) $nP+1-(n-1)+\frac{1}{2}(n-1)(n-2)-$ &c. $= nP + (1-1)^{n-1} - 1$ (since $n-1$ is *even*) $= nP - 1$; and therefore $1.2.3\ldots(n-1)+1$ is a multiple of n.

Upon this result we may remark as follows.

(i) $1.2.3\ldots(n-1) = 1(n-1)\,2(n-2)\ldots\{\frac{1}{2}(n-1)\},\ \{n-\frac{1}{2}(n-1)\}$. Now this is manifestly of the form $nP \pm Q$, where $Q = 1^2.2^2\ldots\{\frac{1}{2}(n-1)\}^2$, and the sign will be + or − according as $\frac{1}{2}(n-1)$ is even or odd, that is, according as it is of the form $2p$ or $2p-1$, or n of the form $4p \pm 1$; therefore, by the Theorem, $nP \pm Q + 1$ is a multiple of $n = nP'$ suppose, and $\therefore Q = n(P' \sim P) \mp 1$, or $1^2.2^2\ldots\{\frac{1}{2}(n-1)\}^2 \pm 1$ is a multiple of n, according as n is of the form $4p+1$ or $4p-1$:

(ii) When n is of the form $4p-1$, since $1^2.2^2\ldots\{\frac{1}{2}(n-1)\}^2 - 1$ is a mult. of n, it follows that one of its two factors, $1.2\ldots\frac{1}{2}(n-1)+1$ and $1.2\ldots\frac{1}{2}(n-1)-1$, is a multiple of n.

Ex. 40.

1. If a No. consisting of two parts be prime to one of them, it will be prime also to the other.

2. If m be a prime No. and a and b integers prime to m, then $a^{m-1} - b^{m-1}$ is div. by m.

3. The square of any No. is of the form $5p$ or $5p \pm 1$, and that of any No. prime to 4 is of the form $4p+1$.

4. Every cube No. is of the form $4p$ or $4p \pm 1$, and every fourth power of the form $5p$ or $5p+1$.

5. Every cube No. is of the form $7p$ or $7p \pm 1$, and every sixth power of the form $7p$ or $7p+1$.

6. If n be a prime No., then $1^{n-1} + 2^{n-1} +$ &c. $+ (n-1)^{n-1}$ is of the form $np - 1$.

7. The diff. of the squares of any two primes, >3, is a multiple of 24.

8. If N be odd and prime to 5, $N^4 - 1$ is divisible by 80.

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199. *To find the No. of divisors of a composite number.*

Let a, b, c, &c. be the prime factors of any No. N, which may therefore be expressed in the form $a^p b^q c^r$ &c.: then the different divisors of N are evidently comprised (and no others) in the terms of the product

$(1 + a + a^2 +$ &c. $a^p)(1 + b + b^2 +$ &c. $b^q)(1 + c + c^2 +$ &c. $c^r)$ &c.,

the No. of terms in which is $(p+1)(q+1)(r+1)$ &c., including as divisors both 1 and N.

Thus $360 = 2^3.3^2.5$, and admits of $4.3.2 = 24$ divisors; $400 = 2^4.5^2$, and admits of $5.3 = 15$ divisors.

From the above we may infer as follows:

(i) The No. of divisors of N, containing a, is evidently the No. of terms in the product $(a + a^2 + \&c.\ a^p)\ (1 + b + \&c.)\ \&c.$, which is $p\ (q + 1)\ (r + 1)\ \&c.$:

(ii) Hence the No. of ways in which N can be divided into two factors is $\frac{1}{2}(p + 1)(q + 1)\ \&c.$, except when N is a *square* No., and then p, q, r, &c. will be all even, and $(p + 1)(q + 1)$ &c. will be an odd No.; here the No. required will be $\frac{1}{2}\{(p + 1)(q + 1)\ \&c. + 1\}$, and the two factors will be equal in one of the products:

(iii) The No. of ways in which N can be resolved into factors *prime to each other*, will be the same as if we wrote a, b, c, &c. for a^p, b^q, c^r, &c. in the above, and will be therefore $\frac{1}{2}.2.2.\ \&c. = 2^{n-1}$, if n be the No. of the prime factors, a, b, c, &c.:

(iv) The *sum* of all the divisors of N is

$$\frac{a^{p+1} - 1}{a - 1} \times \frac{b^{q+1} - 1}{b - 1} \times \frac{c^{r+1} - 1}{c - 1} \times \&c.$$

200. *To find the No. of integers less than N, and prime to it.*

First, let $N = a^p$: then all the Nos. from 1 to N are a^p in number; but, of these, the Nos. $a, 2a, 3a$, &c. $a^{p-1}.a$ are multiples of a, and therefore not prime to N, and these are a^{p-1} in number; therefore the number of Nos. less than N and prime to it is $a^p - a^{p-1} = a^{p-1}(a - 1) = N.\dfrac{a-1}{a}$.

Next, let $N = a^p b^q$: then, as before, among the Nos. less than N, there are $a^{p-1}b^q$ multiples of a; $a^p b^{q-1}$ multiples of b; but among these are some which are common to both sets, viz. those which are multiples of ab, in number $a^{p-1}b^{q-1}$: therefore, subtracting these from one of the former sets, we have on the whole, among the Nos. less than N, $a^{p-1}b^q + a^p b^{q-1} - a^{p-1}b^{q-1}$, which are not prime to N: hence the number of Nos., less than N and prime to it, is $a^p b^q - a^{p-1}b^q - a^p b^{q-1} + a^{p-1}b^{q-1} = a^{p-1}b^{q-1}(a - 1)(b - 1)$ $= N.\dfrac{a-1}{a}.\dfrac{b-1}{b}$: and similarly, if $N = a^p b^q c^r$ &c.

Ex. 1. $100 = 2^2.5^2$: hence the No. of integers, which are less than 100 and prime to it, is $100.\frac{1}{2}.\frac{4}{5} = 40$.

Ex. 2. The No. of forms for primes to mod. 20 is the same as the No. of integers less than 10 and prime to it, viz. $10.\frac{1}{2}.\frac{4}{5} = 4$: these are evidently $20p \pm 1$, $20p \pm 3$, $20p \pm 7$, $20p \pm 9$.

Ex. 41.

1. Find the No. of divisors of 60, 90, 144, 240, 300, and 1000.

2. How many of the above divisors in each case contain the factor 2? and how many contain 4?

3. In how many ways may the first three of the preceding Nos. be divided into two factors? or the last three into factors prime to each other?

4. Find the sum of all the divisors for the first three Nos., in each of the three cases of Ex. 1 and 2.

5. Find the No. of integers less than each of the given Nos. in Ex. 1, and prime to it.

6. How many forms of prime Nos. will there be for each of the six Nos. in Ex. 1? Write them down for 60.

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201. Diophantine Problems. These are so called from Diophantus, who first treated of them about A. D. 360.

The problem is to find such values of x as shall render a given function of x a *square* quantity, which we shall denote by □ or by z^2. It must be observed, however, that many of the questions which occur in this part of Algebra are of considerable difficulty, and require more than ordinary skill and judgment for their solution. The following hints will suffice for most of those which the Student is likely to meet with in actual practice, and will shew the method of treating such questions.

202. I. Given $ax^2 + bx + c = z^2$.

(i) If a be a *square* $= p^2$,

$$\text{put } p^2x^2 + bx + c = \left(px + \frac{m}{n}\right)^2, \text{ whence } x = \frac{m^2 - cn^2}{bn^2 - 2mnp};$$

(ii) If c be a *square* $= p^2$,

$$\text{put } ax^2 + bx + p^2 = \left(\frac{m}{n}x + p\right)^2, \text{ whence } x = \frac{bn^2 - 2mnp}{m^2 - an^2};$$

(iii) If $b^2 - 4ac$ be a *square*, then [65] $ax^2+bx+c = a(x-\alpha)(x-\beta)$, where α, β, are rational;

$$\text{put } (x-\alpha)(x-\beta) = \frac{m^2}{n^2}(x-\alpha)^2, \text{ whence } x = \frac{m^2\alpha - n^2\alpha\beta}{m^2 - an^2}:$$

in which three results m and n may have any values whatever.

(iv) If ax^2+bx+c can be put into the form $(ex+f)^2 + (gx+h)(kx+l)$, put it $= \{(ex+f) + \frac{m}{n}(gx+h)\}^2$, whence x may be easily found.

(v) If any one solution, as $x=r$, can be found in any way, either by rule or by guess, put $x=y+r$; then $a(y+r)^2+b(y+r)+c=z^2$ $=ay^2+(2ar+b)\,y+p^2$, (if $ar^2+br+c=p^2$,) so falling under (ii): or, otherwise, $a(x^2-r^2)+b(x-r)+p^2=z^2=p^2+(x-r)(ax+ar+b)$, so falling under (iv).

N.B. Cases (i) and (ii) include those of $ax^2+bx=z^2$ and $bx+c=z^2$, by supposing $p=0$, that is, by putting $ax^2+bx=\frac{m^2}{n^2}x^2$, $bx+c=\frac{m^2}{n^2}$.

Also $ax^2+bxy+cy^2=z^2$ is included in the above by putting $x=vy$, and dividing by the *square* quantity y^2, which leaves $av^2+bv+c=\square$.

Ex. 1. Given $4x^2-3x-2=\square=\left(2x-\frac{m}{n}\right)^2=4x^2-4\frac{m}{n}x+\frac{m^2}{n^2}$;

$$\therefore x=\frac{m^2+2n^2}{4mn-3n^2}=3, \text{ if } m=1,\ n=1.$$

Ex 2. Given $5x^2+9=\square=\left(\frac{m}{n}x+3\right)^2=\frac{m^2}{n^2}x^2+6\frac{m}{n}x+9$;

$$\therefore x=\frac{6mn}{5n^2-m^2}=12, \text{ if } m=2,\ n=1.$$

Ex. 3. Given $2x^2+x-6=\square=(2x-3)(x+2)=\frac{m^2}{n^2}(x+2)^2$;

$$\therefore x=\frac{2m^2+3n^2}{2n^2-m^2}=5, \text{ if } m=1,\ n=1.$$

Ex. 4. Given $2x^2+x-20=\square=x^2+(x^2+x-20)=x^2+(x-4)(x+5)$

$$=\{x+\frac{m}{n}(x-4)\}^2 \text{ by (iv)}=x^2+2\frac{m}{n}x(x-4)+\frac{m^2}{n^2}(x-4)^2;$$

$\therefore x+5=2\frac{m}{n}x+\frac{m^2}{n^2}(x-4)$, $x=\frac{4m^2+5n^2}{m^2+2mn-n^2}=24$, if $m=1$, $n=2$.

Ex. 5. Given $3x^2-2xy+3y^2=z^2$, or (putting $x=vy$, and dividing by y^2,) $3v^2-2v+3=\square$: since this admits of $v=1$, put $v=u+1$; then $3u^2+4u+4=\square=(\frac{m}{n}u+2)^2$, or $u=\frac{4(n^2-mn)}{m^2-3n^2}$, where we may write $-m$ for m, which only amounts to assuming $(-\frac{m}{n}u+2)^2$ instead of $(\frac{m}{n}u+2)^2$, and then $u=\frac{4(mn+n^2)}{m^2-3n^2}$; if $m=2$, $n=1$, then $u=12$; $\therefore v=13=\frac{x}{y}$, and any pair of values of x and y, taken such that $x=13y$, will be a solution, as $x=13$, $y=1$.

But we may obtain more general values for x and y as follows:

$$\text{we have } u = \frac{4mn + 4n^2}{m^2 - 3n^2}, \quad v = \frac{m^2 + 4mn + n^2}{m^2 - 3n^2} = \frac{x}{y};$$

we may therefore take $x = m^2 + 4mn + n^2$, $y = m^2 - 3n^2$, and give any values we please to m and n: thus if $m = 3$, $n = 1$, then $x = 22$, $y = 6$, &c.

In like manner, if in any case $ax^2 + bxy + cy^2 = z^2$, or $av^2 + bv + c = \square$, admits of solution, we may always find integral values for x and y, by taking for them the numerator and denominator respectively of the general value of v.

Also if any solution $x = r$, $y = s$ be found, then $x = pr$, $y = ps$, or $x = \frac{r}{p}$, $y = \frac{s}{p}$, are manifestly solutions, whatever p may be, since if $ar^2 + brs + cs^2 = \square$, so also

$$ap^2r^2 + bp^2rs + cp^2s^2 = p^2(ar^2 + brs + cs^2) = \square.$$

Hence in the above Ex., since $x = 22$, $y = 6$ is a solution, so also is $x = 44$, $y = 12$, or $x = 11$, $y = 3$, &c.; that is, any pair of *equi-multiples* or *sub-multiples* of the first.

203. The equation $ax^2 + bx = z^2$ may always be solved in *positive integers;* for if $ax^2 + bx = \frac{m^2}{n^2}x^2$, then $x = \frac{bn^2}{m^2 - an^2}$. Now by [166] it is always possible to solve $m^2 - an^2 = 1$ in integers, and, putting in the numerator the value of n thus found, we shall have x a positive integer.

But the equation $ax^2 - bx = z^2$ may always be solved in *integers*, but not always in *positive* integers, since this would require $m^2 - an^2 = -1$, which [166] is not always possible.

Also cases (ii) and (iii) can always be solved in *positive integers:* for the den^r in each case being $m^2 - an^2$, we can put this $= 1$, and solve it in all cases, and then we can take positive or negative values for m or n in the numerator, according to circumstances, so as to get x positive.

And Case v may be treated thus:

$$a(x^2 - r^2) + b(x - r) = z^2 - p^2 = (x - r)\{a(x + r) + b\};$$

put $\quad z + p = \frac{m}{n}(x - r), \quad z - p = \frac{n}{m}\{a(x + r) + b\};$

then, eliminating,

$$x = \frac{(m^2 + an^2)r + 2mnp + bn^2}{m^2 - an^2}, \quad z = \frac{2amnr + (m^2 + an^2)p + bmn}{m^2 - an^2},$$

whence, solving the equation $m^2 - an^2 = 1$, we may get positive integer values for x and z, taking m or n (or p, since it originally enters in the form of p^2,) positive or negative, as we please. And, of course, any value of x thus found may again be substituted for r in the numerators, (retaining the same values of m and n) to find new solutions, and so on.

Ex. If $3v^2 - 2v + 3 = z^2$ (as in Ex. 5), where $r = 1$,

then $\quad 3(v^2 - 1) - 2(v - 1) = z^2 - 4 = (v - 1)(3v + 1)$;

$\therefore\ z + 2 = \dfrac{m}{n}(v - 1),\ \ z - 2 = \dfrac{n}{m}(3v + 1)$, and $v = \dfrac{m^2 + 4mn + n^2}{m^2 - 3n^2}$,
as before.

Here then we must solve $m^2 - 3n^2 = 1$: now $\sqrt{3} = 1 + \dfrac{1}{1+}\,\dfrac{1}{2+}\,\dfrac{1}{1+\&c.}$, and the convergents are 1, $\frac{2}{1}$, $\frac{5}{3}$, $\frac{7}{4}$, &c.;

$$\therefore\ m = 2,\ n = 1,\ \text{or}\ m = 7,\ n = 4,\ \&c.;$$

$\therefore\ v = 13$, or $v = 177$, &c.: and, of course, we might start again, taking $r = 13$, and so on.

N.B. The above method includes the case of $b = 0$ or $ax^2 + c = z^2$, which may be similarly treated, and can always be solved in positive integers, *if one solution, $x = r$, be given.*

Ex. 42.

Find values for x (when possible, in positive integers) which shall render square the following quantities:

1. $4x^2 + 29$. 2. $7x^2 - 5x + 1$. 3. $6x^2 + x - 1$, by (iii).
4. $2x^2 - 1$, by (iv). 5. $2(x^2 - 1)$, by (iii). 6. $2x^2 + 2$, by (iv).
7. $10x^2 + 7x + 1$, by (ii). 8. $5x^2 + 11$, given $r = 1$.
9. $2x^2 - x$, $r = 1$, 10. $6x^2 + x - 1$, $r = 2$. 11. $2x^2 - 1$, $r = 1$.
12. $7x^2 + 2$, given $r = 2$. 13. $6 - 13x + 6x^2$, $r = 2$. 14. $5x^2 - 3x + 7$, $r = 1$.
15. $6(x^2 - x)$, given $r = 3$. 16. $7x^2 - 3$, $r = 1$ or 2.

Find positive integral values of x and y to rationalize the expressions

17. $\sqrt{(11x^2 + 7xy + y^2)}$. 18. $\sqrt{(4x^2 + 5xy + 6y^2)}$.
19. $\sqrt{(6x^2 + 7xy - 3y^2)}$. 20. $\sqrt{(5x^2 - 3xy + 2y^2)}$.

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204. II. (i) Given $ax + b = \square$, $cx + d = \square$, for the same values of x: put $ax + b = z^2$, and substitute the values of x from this in $cx + d = \square$, which will now become of the form $a'z^2 + b' = 0$, and will fall under I.

(ii) Given $ax^2 + bx = \square$ and $cx^2 + dx = \square$; put $\dfrac{1}{y}$ for x; then $a + by = \square$, $c + dy = \square$, which falls under the last case.

Ex. 1. Given $2x - 6 = \square$, $3x + 7 = \square$: from $2x - 6 = z^2$, we get $x = \frac{1}{2}(z^2 + 6)$, whence $3x + 7$ becomes $\frac{3}{2}(z^2 + 6) + 7 = \square$, or (mult. by 4, a *square* No.) $6z^2 + 64 = \square = \left(\frac{m}{n} z + 8\right)^2 = \frac{m^2}{n^2} z^2 + 16 \frac{m}{n} z + 64$;

$\therefore z = \dfrac{16mn}{6n^2 - m^2} = 16$ if $m = 2$, $n = 1$, and $\therefore x = \frac{1}{2}(z^2 + 6) = 131$.

Ex. 2. Given $x + 3x^2 = \square$, $9x + 5x^2 = \square$: for x put $\dfrac{1}{y}$, then $\dfrac{1}{y} + \dfrac{3}{y^2} = \square$, or (mult. by y^2) $y + 3 = \square = z^2$, and $y = z^2 - 3$;

$$\text{so } 9y + 5 = 9z^2 - 22 = \square = \left(3z - \frac{m}{n}\right)^2 = 9z^2 - 6\frac{m}{n} z + \frac{m^2}{n^2};$$

$\therefore z = \dfrac{m^2 + 22n^2}{6mn} = \frac{13}{6}$, if $m = 2$, $n = 1$, and $y = z^2 - 3 = \frac{61}{36}$; $\therefore x = \frac{36}{61}$.

205. III. (i). Given $ax + by = \square = p^2$, $cx + dy = \square = q^2$, then $x = \dfrac{dp^2 - bq^2}{ad - bc}$, $y = \dfrac{aq^2 - cp^2}{ad - bc}$, and if p and q be taken any multiples of $ad - bc$, such as $p = m(ad - bc)$, $q = n(ad - bc)$, then x and y will be obtained in integers.

(ii) Given $ax + by = \square$, $cx + dy = \square$, $ex + fy = \square$: put $ax + by = p^2$, $cx + dy = q^2$, whence, substituting for x and y, $ex + fy = a'p^2 + b'q^2 = \square$, or, putting $q^2 = p^2z^2$, and dividing by the *square* p^2, $a' + b'z^2 = \square$.

Ex. 1. $x + 2y = \square = p^2$, $3x + 4y = \square = q^2$; $\therefore x = q^2 - 2p^2$, $y = \frac{1}{2}(3p^2 - q^2)$, whence, if $p = 2$, $q = 3$, we have $x = 1$, $y = 1\frac{1}{2}$; or, if $p = 3$, $q = 5$, then $x = 7$, $y = 1$; or if $p = 2m$, $q = 2n$, then $x = 4(m^2 - 2n^2)$, $y = 2(3m^2 - n^2)$.

Ex. 2. $x + 2y = \square$, $3x + 4y = \square$, $7x + 6y = \square$: taking the general values for x and y found in Ex. 1 from the first two equations, we have $7x + 6y = 4q^2 - 5p^2 = \square$, or (putting $q = pz$, and div. by p^2) $4z^2 - 5 = \square = \left(2z - \frac{m}{n}\right)^2 = 4z^2 - 4\frac{m}{n} z + \frac{m^2}{n^2}$; $\therefore z = \dfrac{m^2 + 5n^2}{4mn} = \frac{3}{2}$, if $m = 1$, $n = 1$; hence $q = \frac{3}{2}p$, and $x = q^2 - 2p^2 = \frac{1}{4}p^2$, $y = \frac{1}{2}(3p^2 - q^2) = \frac{3}{8}p^2$; if $p = 4$, then $x = 4$, $y = 6$.

206. The above solutions in (I, II, III) are all *general* ones, by means of which we may obtain an unlimited No. of different values for x, all satisfying the conditions of the Problem, though they may not all be integral: but in the following (excepting the first) the Solutions are limited, generally to one only, the values of m and n being fixed by the steps taken.

207. IV. Given $ax^3 + bx^2 + cx + d = \square$:

(i) If $c = 0$, $d = 0$, put $ax^3 + bx^2 = m^2x^2$;

(ii) If $d = p^2$, put $ax^3 + \&c. = (mx + p)^2$, and then put $2mp = c$.

(iii) In no other case can rules be given, except when a solution $x = r$ can be found in any way, when by putting $x = y + r$, the case will fall under (ii).

Ex. 1. $2x^3 + 3x^2 = \square = m^2x^2$; $\therefore\ x = \frac{1}{2}(m^2 - 3) = 3$, if $m = 3$.

Ex. 2. $2x^3 + 3x^2 + 5x + 4 = (mx + 2)^2 = m^2x^2 + 4mx + 4$; put $4m = 5$, then $2x^3 + 3x^2 = m^2x^2$, or $2x + 3 = \frac{25}{16}$, and $x = -\frac{23}{32}$, which solution we might put $= r$, and employ as in Ex. 3, to find another.

Ex. 3. $2x^3 - 3x^2 + 5x - 5 = \square$ has a solution $x = 2$: put $x = y + 2$; then $2y^3 + 3y^2 + 17y + 9 = \square = (my + 3)^2 = m^2y^2 + 6my + 9$; put $6m = 17$; $\therefore\ 2y + 3 = m^2 = \frac{289}{36}$, and $y = \frac{181}{72}$; $\therefore\ x = y + 2 = 4\frac{37}{72}$.

208. V. Given $ax^4 + bx^3 + cx^2 + dx + e = \square$.

(i) If $a = p^2$, put $ax^4 + \&c. = (px^2 + mx + n)^2$, and then put $2mp = b$, $2np + m^2 = c$, which gives m and n;

(ii) If $e = p^2$, put $ax^4 + \&c. = (mx^2 + nx + p)^2$, and then put $2np = d$, $2mp + n^2 = c$;

(iii) If $a = p^2$, $e = q^2$, put $ax^4 + \&c. = (px^2 + mx + q)^2$, and then put either $2mp = b$ or $2mq = d$, by which we get two different solutions; or put $ax^4 + \&c. = (px^2 + mx - q)^2$, and proceed as before, by which we get two other solutions, which, however, we may obtain from the former by merely changing the sign of q. Also, since this case comes under both (i) and (ii), we may by the methods there given obtain two other solutions, that is, *six* in all.

(iv) Lastly, if a solution $x = r$ is known, we may put $x = y + r$, as before.

Ex. $4x^4 - 3x^3 + 2x^2 - 2x + 1 = \square$: this comes under (iii), but will enable us to illustrate also (i) and (ii).

By (i) put $4x^4 - 3x^3 + 2x^2 - 2x + 1 = (2x^2 + mx + n)^2$

$$= 4x^4 + 4mx^3 + (4n + m^2)\,x^2 + 2mnx + n^2;$$

put $4m = -3$, $4n + m^2 = 2$; $\therefore\ m = -\frac{3}{4}$, $n = \frac{23}{64}$;

and now $-2x + 1 = 2mnx + n^2 = -\frac{69}{128}x + \frac{529}{4096}$; $\therefore\ x = \frac{3567}{5984}$.

By (ii) put $4x^4 - 3x^3 + 2x^2 - 2x + 1 = (mx^2 + nx + 1)^2$

$$= m^2x^4 + 2mnx^3 + (2m + n^2)\,x + 2nx + 1;$$

put $2n = -2$, $2m + n^2 = 2$; $\therefore\ n = -1$, $m = \frac{1}{2}$;

and now $4x^4 - 3x^3 = m^2x^4 + 2mnx^3$, or $4x - 3 = \frac{1}{4}x - 1$; $\therefore\ x = \frac{8}{15}$.

By (iii) put $4x^4 - 3x^3 + 2x^2 - 2x + 1 = (2x^2 + mx \pm 1)^2$
$$= 4x^4 + 4mx^3 + (m^2 \pm 4)\,x^2 \pm 2mx + 1;$$
put $4m = -3$; $\therefore m = -\frac{3}{4}$, and $2x^2 - 2x = (m^2 \pm 4)\,x^2 \pm 2mx$;
$\therefore$ with upper signs, $2x - 2 = (\frac{9}{16} + 4)\,x - \frac{3}{2}$, whence $x = -\frac{8}{41}$,
with lower signs, $2x - 2 = (\frac{9}{16} - 4)\,x + \frac{3}{2}$, whence $x = \frac{56}{87}$;
or put $\pm 2m = -2$; $\therefore m = -1$ or $+1$, according as we take the upper or lower signs;
and $-3x^3 + 2x^2 = 4mx^3 + (m^2 \pm 4)\,x^2$, or $-3x + 2 = 4mx + m^2 \pm 4$;
$\therefore$ with upper signs, $-3x + 2 = -4x + 5$, whence $x = 3$,
with lower signs, $-3x + 2 = 4x - 3$, whence $x = \frac{5}{7}$.
Lastly, since $x = 3$ is a solution, put $x = y + 3$;
$$\text{then } 4y^4 + 45y^3 + 191y^2 + 361y + 256 = \square,$$
which is of the same form as the given one, and, being similarly treated, would yield other solutions.

209. VI. Given $ax^3 + bx^2 + cx + d =$ a perfect *cube:*
(i) If $a = p^3$, put $ax^3 + \&c. = (px + m)^3$, and then put $3mp^2 = b$;
(ii) If $d = p^3$, put $ax^3 + \&c. = (mx + p)^3$, and then put $3mp^2 = c$;
(iii) If $a = p^3$, $d = q^3$, put $ax^3 + \&c. = (px + q)^3$;
(iv) If $x = r$ is a solution, put $x = y + r$.

Ex. 43.

Solve the Diophantine equations

1. $x + 11 = \square$, $x - 13 = \square$.
2. $x^2 + x = \square$, $x^2 - x = \square$.
3. $2x + y = \square$, $x - 2y = \square$.
4. $2x^2 + x = \square$, $3x^2 + 2x = \square$
5. $11x^3 + 3x^2 = z^2$.
6. $x^3 - 2x^2 + 2x + 1 = z^3$.
7. $x^3 - 3x^2 + 3 = z^3$, $r = 1$.
8. $1 - 2x + 3x^2 - 3x^3 + 4x^4 = z^2$.
9. $x^3 - 2x^2 + 3 = z^3$.
10. $3x^2 + 12x - 8 = z^3$.
11. $7x^3 + 1 = z^3$, $r = 1$.
12. $5 - 3x^3 = z^3$, $r = -1$.

PROB. 1. Find the general values of x, y, and z, in $ax^2 + y^2 = z^2$.

Here put $x = vy$, then $av^2 + 1 = \square = (\frac{m}{n}v - 1)^2$, $v = \dfrac{2mn}{m^2 - an^2} = \dfrac{x}{y}$:
$$\therefore x = 2mn,\quad y = m^2 - an^2,\quad z = m^2 + an^2.$$
If $a = 1$, then $x^2 + y^2 = z^2$, and $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$.

Hence the fraction $\dfrac{2mn}{m^2 - n^2}$ is such that the sum of the squares of its num^r and den^r is always a square. For m write $n + 1$; then this fraction becomes $\dfrac{2n^2 + 2n}{2n + 1} = n + \dfrac{n}{2n + 1}$, from which, by giving

n the successive values 1, 2, 3, &c., we get the following curious series of fractions, $1\frac{1}{3}$, $2\frac{2}{5}$, $3\frac{3}{7}$, $4\frac{4}{9}$, &c., each of which possesses the above-mentioned property.

PROB. 2. Find three square integers in A. P.

Let x^2, y^2, z^2 be the numbers; then $x^2+z^2=2y^2$, or if $x=p-q$, $z=p+q$, then $p^2+q^2=y^2$: now by Prob. 1 this is satisfied by $p=m^2-n^2$, $q=2mn$, and $y=m^2+n^2$; $\therefore$ the three quantities are $(m^2-n^2-2mn)^2$, $(m^2+n^2)^2$, and $(m^2-n^2+2mn)^2$, where m and n may have any values. If $m=n=1$, these are 1, 25, 49.

PROB. 3. Find a number x such that x^2+ax and x^2-ax may both be squares.

Let $x^2+ax=p^2x^2$; $\therefore\ x=\dfrac{a}{p^2-1}$, and $x^2-ax=\dfrac{a^2}{(p^2-1)^2}-\dfrac{a^2}{p^2-1}=\square$,

$$\text{whence } 1-(p^2-1)=2-p^2=\square=z^2 \text{ suppose}:$$

$$\therefore\ z^2-1=1-p^2:\ \text{let } z+1=\frac{m}{n}(1-p),\ \ z-1=\frac{n}{m}(1+p);$$

$$\therefore\ p=\frac{m^2-2mn-n^2}{m^2+n^2},\ \text{and } p^2-1=\frac{4mn(n^2-m^2)}{(m^2+n^2)^2}:\ \therefore\ x=\frac{(n^2+m^2)^2a}{4mn(n^2-m^2)}.$$

If $m=1$, $n=2$, then $x=\frac{25}{24}a$, which evidently satisfies the question.

PROB. 4. Find three integers in A. P., so that the sum of every two may be a square.

Let there be x, $x+y$, $x+2y$; then

$$2x+y=\square,\quad 2x+2y=\square,\quad 2x+3y=\square:$$

$$\text{let } 2x+y=p^2,\quad 2x+2y=q^2;\quad \therefore\ 2x=2p^2-q^2,\quad y=q^2-p^2,$$

$$\text{and } 2x+3y=2q^2-p^2=\square:\ \text{hence } 2v^2-1=z^2,\ \text{or } z^2-v^2=v^2-1:$$

$$\text{let } z+v=\frac{m}{n}(v-1),\quad z-v=\frac{n}{m}(v+1);\quad \therefore\ v=\frac{m^2+n^2}{m^2-2mn-n^2}=\frac{q}{p};$$

$$\therefore\ q=m^2+n^2,\ p=m^2-2mn-n^2,\ q^2-p^2=(q+p)(q-p)=4mn(m^2-n^2):$$

$$\therefore\ 2x=p^2+(p^2-q^2)=(m^2+n^2)^2-8mn(m^2-n^2),\quad y=4mn(m^2-n^2),$$

where, in order that x may be integral, we must have m and n both even or both odd. If $m=9$, $n=1$, $x=482$, $y=2880$, and the numbers required are 482, 3362, 6242.

PROB. 5. Find two numbers such that if to each and also to their sum a given square a^2 be added, the three sums shall be squares.

Let the numbers be x^2-a^2, y^2-a^2, by which assumption the first two conditions are satisfied: then $x^2+y^2-a^2=z^2$, or $z^2-y^2=x^2-a^2$:

let $z+y=\frac{m}{n}(x-a)$, $z-y=\frac{n}{m}(x+a)$; then $x=\frac{(m^2-n^2)a+2mnz}{m^2+n^2}$, $y=\frac{(m^2-n^2)z-2mna}{m^2+n^2}$, where z may have any value.

If $m=2$, $n=1$, then $x=\frac{1}{5}(3a+4z)$, $y=\frac{1}{5}(3z-4a)$, and if $a=2$, $z=11$, then $x=10$, $y=5$, and the numbers are $x^2-a^2=96$, $y^2-a^2=21$.

PROB. 6. Find three square integers such that their sum may be a given square.

Let $x^2+y^2+z^2=a^2$; then this is satisfied by putting $y^2=2xz$;

$$\therefore x+z=a, \text{ and } y^2=2xz=2ax-2x^2=\square=\frac{m^2}{n^2}x^2 \text{ suppose:}$$

$$\therefore x=\frac{2n^2a}{m^2+2n^2},\quad z=\frac{m^2a}{m^2+2n^2},\quad y=\frac{2mna}{m^2+2n^2}.$$

If $m=1$, $n=1$, and $a=3$, then $x=2$, $y=2$, $z=1$, and $2^2+2^2+1^2=3^2$; if $m=1$, $n=2$, and $a=9$, then $x=8$, $y=4$, $z=1$, and $8^2+4^2+1^2=9^2$.

Ex. 44.

1. Find a number x such that $x+1$ and $x-1$ shall be squares.

2. What is the least number of terms of the series 1, 2, 3, &c., whose sum is a square?

3. Find two squares, whose difference shall be a; and apply the result to find integral squares, when $a=15$, and when $a=16$.

4. Find the least two integers, whose difference is a cube and sum a square.

5. Find the least two integers such that, if the square of each be added to their product, the sums shall be squares.

6. Find two squares such that their difference shall be a given square; and apply the result to find two integers, whose difference shall be 441.

7. Find the least two integers whose sum and diff. are squares.

8. Find two numbers such that, if their product be added to the sum of their squares, the result shall be a square.

9. Divide a given square into two squares. Ex. 225.

10. Find the general values of x, y, and z, in $x^2-ay^2=z^2$.

11. Find two numbers such that, if each be added to their product, the sums shall be squares.

12. Find general expressions for converting a number, which is the sum of two squares a^2 and b^2, into the sum of two other squares.

CHAPTER XII.

PROBABILITIES.

210. If an event *may* happen in n ways, each equally possible, the probability that it *will* happen in any given one of them is properly represented by $\frac{1}{n}$, if we represent *certainty* by *unity*.

For let x represent this probability. Then, since the event must happen in some one or other of the n ways, the sum of all their separate probabilities must be certainty; hence $nx = 1$, and $x = \frac{1}{n}$.

So also the probability of its happening in any one out of m specified ways is the sum of m such separate probabilities, or $\frac{m}{n}$.

Ex. If there were 20 tickets distributed to as many persons, the holder of one of them to be entitled to a prize, then, since it is, *a priori*, equally possible that any one of them may hold the fortunate ticket, the probability of success for each person is $\frac{1}{20}$; and if any one person held *seven* such tickets, he would have the same probability of success as seven different persons, with their separate tickets, would have between them, that is, $\frac{7}{20}$.

The fraction in each case is called the *Mathematical chance*, or simply the *chance* of the corresponding event happening, and represents, as we have seen, the probability of this, on the supposition that *certainty* is represented by *unity*.

211. If there are $a + b$ occurrences, all equally possible, a of them favourable to a certain event and b unfavourable, the chance of that event happening is $\frac{a}{a+b}$, and of its failing to happen, $\frac{b}{a+b}$.

For the chance that any one of the $a+b$ occurrences will happen is $\frac{1}{a+b}$; and therefore $\frac{a}{a+b}$, $\frac{b}{a+b}$, are respectively the chances that some one or other of the a favourable ones or of the b unfavourable ones will happen; the former then is the chance of the event itself happening, the latter of its failing to happen.

Or thus: It is plain, from the hyp., that the chance of the event's happening : chance of its failing :: $a : b$; ∴ chance of happening : chance of happening + chance of failing :: $a : a + b$;
but chance of happening + chance of failing = certainty = 1;

$$\therefore \text{ chance of happening} = \frac{a}{a+b}, \text{ and chance of failing} = \frac{b}{a+b}.$$

In this case the odds are said to be a to b upon the event, the odds being in the ratio of the chances for and against it. If we denote the former chance by p and the latter (or $1-p$) by q, then the odds will be $p : q$ on the event, *in its favour* or *against it*, according as $p >$ or $< q$. If $a = b$, the odds are *even*, and $p = \frac{1}{2} = q$.

If we have the chance of an event in the form of a fraction, we may obtain the odds at once by taking the num^r from the den^r: thus, if the given chance be $\frac{2}{5}$, the odds are 2 : 3 against the event, this being the ratio of the chances $\frac{2}{5}$, $\frac{3}{5}$, for and against it.

Def. The chance then of any event is represented mathematically by the fraction, whose *den^r* is the No. of *possible* occurrences and *num^r* the No. of them *favourable* to the event in question.

Ex. 1. If there be 10 balls in a bag and *one* be drawn, the chance of its being one of three marked ones is $\frac{3}{10}$, there being here 10 *possible* occurrences and 3 *favourable*: if *two* be drawn, the chance of their being two marked ones is $\frac{1}{45}$; for there are here 45 *possible* occurrences (since the 10 balls may be drawn, 2 together, in 45 ways) and 1 only is *favourable*: the chance of one only of the two drawn being one of two marked ones is $\frac{16}{45}$; for each of the two marked ones may be drawn with either of the 8 others, making altogether 16 *favourable* cases.

Ex. 2. In a bag are four white and six black balls: find the chance that, out of five drawn, two and two only shall be white.

There are here 252 *possible* occurrences, since 10 balls can be drawn, 5 together, in 252 ways; and for the *favourable* ones, each two of the four white balls (which can be taken, 2 together, in 6 ways) may be drawn with any three of the six black balls (which can be taken, 3 together, in 20 ways), making altogether $6 \times 20 = 120$ favourable occurrences: hence the chance required is $\frac{120}{252} = \frac{10}{21}$.

Ex. 3. Find the chance in Ex. 2 of two at least being white.

Here we must find the chances that two only, three only, and four, out of the five drawn, shall be white: these will be found to

be respectively $\frac{10}{21}$, $\frac{5}{21}$, $\frac{1}{42}$, and their sum, $\frac{31}{42}$, is the chance required.

212. In all the above and similar instances, if we subtract from *unity* the chance we have obtained, we shall get the chance that the event in question (A suppose) will *not* happen, or, which is the same thing, that some other, which may be called *complementary*, event (B suppose) will happen.

Thus, in Ex. 3, the chance of *not* drawing as many as two white balls, or the chance of drawing at least four black balls, is $1 - \frac{31}{42} = \frac{11}{42}$, and the odds will be 11 : 31 against this event.

This may be often applied to simplify the calculation of chances; since it may be easier to estimate the chance for the complementary event, than for the given one. Thus in Ex. 3, the chance of the complementary event is the sum of only *two* chances, viz. that of drawing four black balls ($= \frac{5}{21}$) and that of drawing five ($= \frac{1}{42}$), whereas we had before to find the sum of *three* chances; and now $\frac{5}{21} + \frac{1}{42} = \frac{11}{42}$, $1 - \frac{11}{42} = \frac{31}{42}$, as before.

213. In like manner, if the No. of equally possible occurrences be $a+b+c$, and a of them be favourable to an event A, b to B, and c to C; the chance that A, B, or C will happen respectively is $\frac{a}{a+b+c}$, $\frac{b}{a+b+c}$, $\frac{c}{a+b+c}$, which if we denote by p, q, r, then $p+q+r=1$.

The complementary chance to any one of these, as A, is the sum of the other two, since $1-p=q+r$, and it is plain that if A does not happen, either B or C *will*. Hence the odds upon A are $a : b+c$, or $p : q+r$; and similarly for any No. of events.

Ex. 45.

1. The odds being $1 : 2\frac{1}{2}$, $2\frac{1}{2} : 1\frac{1}{2}$, $2\frac{1}{2} : 3\frac{1}{3}$, find the corresponding chances, and compare them.

2. One of two events must happen: given that the chance of the one is $\frac{2}{3}$rds of that of the other, find the odds upon the first.

3. On the same hypothesis, given that the odds upon the one are $2\frac{1}{4}$ of those upon the other, find the chances of each.

4. There are three events, A, B, C, one of which must happen: the odds are 3 : 8 on A, and 2 : 5 on B; find the odds on C.

5. In a bag are 3 white and 5 red balls: find the chances that, *one* being drawn, it shall be (i) white or (ii) red; or *two* being

drawn, they shall be (i) both white, (ii) both red, or (iii) white and red.

6. In a bag are 3 white, 4 red, and 5 black balls; if *one* be drawn, find the chance of its being (i) white, (ii) red, or (iii) black; if *two* be drawn, of their being (i) white and red, (ii) both black, or (iii) one at least red; if three be drawn, of their being (i) one of each colour, (ii) two of them black, or (iii) one of them white.

7. In a bag are 10 balls, of which 4 are drawn: find the chance that there shall be among them (i) two marked ones, (ii) two only, out of three marked ones, (iii) three only out of five marked ones, (iv) two at least out of six marked ones.

8. A general orders two men, by lot, out of 100 mutineers, to be shot; the real leaders of the mutiny being 10, find the chance that (i) one, or (ii) two of them will be taken.

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214. If two events are independent of each other, and the chance of the first happening is p_1, and of the second p_2, then the chance of *both* happening will be p_1p_2.

For, as before, let $p_1 = \dfrac{a_1}{a_1+b_1}$, $p_2 = \dfrac{a_2}{a_2+b_2}$: then since any one of the a_1+b_1 occurrences may happen in company with any one of the a_2+b_2, there will be $(a_1+b_1)(a_2+b_2)$ ways in which it is *possible* for these occurrences, one for each event, to happen together, and of these a_1a_2 will be favourable to the happening of both events; hence the chance of both happening is

$$\frac{a_1a_2}{(a_1+b_1)(a_2+b_2)} = p_1p_2.$$

Such chances are called *contingent* chances. And in the same way exactly it may be shewn that if q_1 (or $1-p_1$) be the chance of *failure*, or the complementary chance, of the first event, and q_2 (or $1-p_2$) of the second, then p_1q_2 will be the chance of the first happening and second failing, p_2q_1 of the first failing and second happening, q_1q_2 of both failing.

The same mode of reasoning will apply to any No. of contingent chances. Thus, if there be n independent events, the chance of all happening is $p_1p_2\ldots p_n$, and of all failing, $q_1q_2\ldots q_n$, of all happening but the last, $p_1p_2\ldots p_{n-1}q_n$, of all failing but the first, $p_1q_2\ldots q_n$, &c.

215. In like manner the chance of the *same* event happening each time upon n trials is p^n; the chance of its happening $n-1$ times and failing once in any *given* order is $p^{n-1}q$, or in *any* order whatever, $np^{n-1}q$ (since the failure may occur in either of the n trials); the chance of its happening $n-2$ times and failing twice, in *given* order, is $p^{n-2}q^2$, or, in *any* order, $\frac{1}{2}n(n-1)\,p^{n-2}q^2$, (since there are $\frac{1}{2}n(n-1)$ ways in which two out of the n trials may be taken as being those of failure); and, generally, the chance of its happening in $n-r$ ways, and failing in r ways, in *given* order, is $p^{n-r}q^r$, or, in *any* order, $C_r p^{n-r}q^r$, the coefficient being the No. of ways in which the n trials may be combined, r together, as those of failure. It is plain that the chances in the above are the successive terms of the expansion of $(p+q)^n$, taken without or with their coefficients, according as the *order* of the happening or failure of the event is specified or not.

COR. 1. The chance of its happening at least r times in n trials is

$$p^n + np^{n-1}q + \tfrac{1}{2}n(n-1)\,p^{n-2}q^2 + \&c. + C_{n-r}p^r q^{n-r}.$$

COR. 2. Writing $p = \dfrac{a}{a+b}$, $q = \dfrac{b}{a+b}$, the above chances become

$$\frac{a^n}{(a+b)^n},\ \frac{na^{n-1}b}{(a+b)^n},\ \frac{n(n-1)}{1.2}\cdot\frac{a^{n-2}b^2}{(a+b)^n},\ \&c.$$

216. In like manner, if there be three events A, B, C, one of which must happen and the others fail at each trial, with chances p, q, r respectively as in [213], then the chance that A will happen α times, B β times, and C γ times, in n trials, (where $\alpha+\beta+\gamma=n$) will be that term of the expansion of $(p+q+r)^n$, which involves $p^\alpha q^\beta r^\gamma$, with or without its coefficient, according as the events are to happen in any order whatever, or in some one specified order. If we replace p, q, r by their values, the chance will be expressed by a fraction, whose numerator is a similar function of a, b, c, and denominator $(a+b+c)^n$.

And the same reasoning is plainly applicable to any No. of events.

The student will easily perceive how the results, obtained under the Binomial Theorem, may be applied to Chances; as, for instance, to find the *most likely* result, upon n trials of the same event, we must find the greatest term of $(p+q)^n$, or of $(a+b)^n$.

EX. 1. When a coin is tossed, the chance of its falling heads (or tails) once, twice, thrice, &c. successively, is $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c.

Ex. 2. A bag contains 3 white and 7 black balls: the chance of drawing white the first time is $\frac{3}{10}$; supposing this done, there will be only 9 balls in the bag, of which two are white, and the chance of drawing white again will be $\frac{2}{9}$; similarly, the chance of drawing white the third time is $\frac{1}{8}$; hence the chance of drawing white in each of the first three trials is $\frac{3}{10} \times \frac{2}{9} \times \frac{1}{8} = \frac{1}{120}$.

So the chance of drawing two white and one black in three trials will be $3 \times \frac{7}{120} = \frac{7}{40}$; for it may be done in *three* ways, *white, white, black*, or *white, black, white*, or *black, white, white*, the chance for the first of which is $\frac{3}{10} \times \frac{2}{9} \times \frac{7}{8} = \frac{7}{120}$, and similarly for the other two.

Ex. 3. Taking the same case as in Ex. 2, but *replacing* in the bag the ball drawn each time, the chance of drawing white three times successively is $p^3 = (\frac{3}{10})^3 = \frac{27}{1000}$; that of drawing two white and one black, *with* or *without* regard to order, is p^2q or $3p^2q$, respectively, $= \frac{63}{1000}$ or $\frac{189}{1000}$; &c.

Ex. 4. In *ten* trials, as in Ex. 3, the most likely event is that for which $C_r p^{n-r} q^r$ is greatest, when $n = 10$, $p = \frac{3}{10}$, $q = \frac{7}{10}$; this will be found to be when $r = 7$; hence the most likely result in ten trials is that we shall have drawn 3 white and 7 black balls.

Ex. 5. Shew that it is probable that a person will throw an *ace* at least once in four throws with a single die.

The chance of *not* throwing an ace in each of four trials is q^4 (where $q = \frac{5}{6}$) $= \frac{625}{1296}$; therefore chance of throwing it $= \frac{671}{1296} > \frac{1}{2}$.

Ex. 6. In a bag there are 3 white, 4 red, and 5 black balls: the chance that in 6 trials (replacing) there shall have been drawn two of each colour = term involving $p^2q^2r^2$ in $(p + q + r)^6$, $= 90p^2q^2r^2 = 90 \times \frac{1}{16} \times \frac{1}{9} \times \frac{25}{144} = \frac{125}{1152}$.

Ex. 46.

1. Find the odds that, on tossing a shilling thrice, it will fall (i) head and two tails without respect to order, (ii) head, tail, head.

2. Find the odds that, if a shilling be tossed four times, it will fall two heads and two tails, sooner than four heads.

3. If two shillings are tossed three times, find the odds that they will fall (i) five heads and a tail, (ii) two heads and four tails.

4. If a shilling be tossed 5 times, how many times is it most likely to fall heads, and what is the chance of this?

5. A bag contains 4 white and 6 red balls; A, B, C draw each a ball in order, replacing: find the chance that they have drawn (i) each white, (ii) A and B white, C red, (iii) two white, one red. Find also the same if the balls are *not* replaced.

6. A draws four times from a bag, containing 2 white and 8 black balls, replacing; find the chance that he will have drawn (i) two white, two black, (ii) not less than two white, (iii) not more than two white, (iv) one white, three black.

7. Find the most probable event in (6). What would it have been if there had been 4 white and 6 black balls?

8. What is the chance that if a shilling be tossed five times, it will fall heads either twice or else three times? Find the same chance, if it be tossed six times.

9. There are 9 balls in a bag, 5 red, 4 white; shew that the most probable event in 5 drawings is the same, whether the balls be replaced or not. Shew the same also in 7 drawings, and that the ratio of the two former chances : ratio of the two latter :: 243 : 70.

10. What is the chance of throwing an ace (i) three times exactly, (ii) not less than three times, (iii) not more than three times, in five throws with a single die?

11. In a bag are 3 white, 5 red, and 7 black balls, and a person draws three times, replacing; find the chance that he will have drawn (i) a ball of each colour, (ii) two white, one red, (iii) three red, (iv) two red, one black.

12. On the same supposition, if he draw five times, find the chances of his drawing (i) three white, one red, one black, (ii) three red, two black, (iii) one white, four red, (iv) one black, four red.

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217. If p be a person's chance of success in respect of any event, and M the amount that will accrue to him, if successful, his *expectation*, or the value of his hope, is Mp.

For let $p = \dfrac{a}{a+b}$; then if $a+b$ persons were equally interested in the event, each depending upon one of the $a+b$ possible occurrences from which it may follow, and if x were the expectation of each, the sum of all their expectations should be equal to the whole gain, or $(a+b)x = M$; hence $x = \dfrac{M}{a+b}$, and the exp. of one, who depends upon a of those occurrences, will be $ax = \dfrac{Ma}{a+b} = Mp$.

Of course, a *loss* will have to be reckoned as a *negative* gain.

Ex. 1. *A* backs one event and *B* another, the respective chances being p and q; what would be a *fair* bet between them?

Let $a : b$ be the bet; then *A*'s exp. $= pb - qa =$ *B*'s exp. $= qa - pb$; $\therefore pb = qa$, and $a : b :: p : q$, that is, in the ratio of the chances. In fact, the expectations in this case ought to be each *zero*.

If bets were constructed on this principle, there would be no *unfairness* in them: but it should never be forgotten that the immorality of gambling consists, not merely in the dishonesty which so often accompanies it, nor in the selfish tempers which it generally tends to foster; but especially in its generating a craving for unwholesome excitement, which, while it is indulged, most certainly impedes the formation of a true and manly character, and, where it becomes habitual, effectually destroys it.

Ex. 2. In a bag are a guinea, a sovereign, and three shillings; *A* is to draw (i) once, (ii) twice: find his expectation in each case.

The whole sum to be drawn is 44*s*; if he drew five coins, he would draw the whole, and his exp. would therefore be 44*s*: hence (i) if he draws once, his expectation is $\frac{1}{5}$ of 44*s* = $8\frac{4}{5}$*s*;

and (ii) if he draws twice, his expectation is double of this = $17\frac{3}{5}$*s*; and so on, in proportion, if he draws again.

Ex. 3. *A* draws five times (replacing) from a bag in which there are 3 white and 7 black balls; every time he draws a white ball he is to receive a shilling, and every time he draws a black ball to pay 6*d*: what is his expectation?

His expectation each time will be $\frac{3}{10} \times 1s - \frac{7}{10} \times 6d = -\frac{1}{20}s$; hence in five trials he may expect a *loss* of 3*d*.

Ex. 4. In a bag are two red and three white balls: *A* is to draw a ball and then *B*, and so on in order, the stake (10*s*) to be won by whichever draws a white ball first: how much should each stake of the 10*s*?

A's chance of drawing white in his *first* trial is $\frac{3}{5}$: the chance of *B having* a first trial is that of *A* drawing red or $\frac{2}{5}$, and then (since there would be now one red, three white) the chance of *B*'s drawing white would be $\frac{3}{4}$; therefore, upon the whole, *B*'s chance of drawing white is $\frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$, and of drawing red $\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$: the chance of *A having* a second trial is that of *B* drawing red or $\frac{1}{10}$, and then (since there would be now three white only) his chance of drawing white would be *certainty* or 1, after which *B* cannot have another trial: hence *A*'s chance altogether is $\frac{3}{5} + \frac{1}{10} = \frac{7}{10}$, and *B*'s $\frac{3}{10}$; therefore their stakes should be *A*'s 7*s*, *B*'s 3*s*.

Ex. 47.

1. A person throws a common die, and is to receive as many pence as the point he throws: what is his expectation? what with two dice?

2. A person throws two dice, receiving as above, except that when he throws *doublets*, he is to receive in pence the *square* of the point: what is his expectation? what is it also, if he is only to receive as above upon the doublets, but to *pay* upon the other points?

3. In a bag are two white and five red balls: *A* is to draw a ball, receiving 1*s* if it be a white ball, and paying 3*d* if it be red: find his expectation, and find also what it would have been, if they had been five white balls and two red?

4. On each of the two suppositions in (3), find *B*'s expectation, who is to draw twice, (i) replacing, (ii) not replacing.

5. A shilling is thrown five times: *A* is to have it, if it fall heads three times and tails twice, and *B*, if it fall heads more than three times: if neither of these events happen, it is to be divided equally between them: what are their expectations?

6. From a bag, containing a sovereign and five shillings, *A* is to draw (i) once, (ii) twice, (iii) thrice: find his exp. in each case.

7. From a bag containing two guineas, three sovereigns, and three shillings, *A* is to draw one coin and *B* three, and *A*, *B*, and *C* are to divide equally the remainder: what are their expectations?

8. From the bag in (7) (i) *A* and *B*, (ii) *A*, *B*, *C*, (iii) *A*, *B*, *C*, *D* are to draw successively till all the coins are drawn: what are their expectations?

9. *A* and *B* draw from a bag in which are three white and three black balls, for a sum of 10*s*, to be received by him who first draws a white ball: *A* has the first draw: what are their expectations, (i) replacing, or (ii) not?

10. *A*, *B*, and *C*, staking each 5*s*, draw from a bag, in which there are 4 white and 6 black balls, each drawing in order, and the whole is to be received by him who first draws a white ball: what are their expectations, (i) replacing, or (ii) not?

11. There are 10 balls in a bag, two of which are marked: a person pays down 6*d*, and is allowed to draw 3 balls, receiving 2*s* for each marked ball he draws, and paying 3*d* for every other: what is his expectation, (i) replacing, or (ii) not?

12. *A*, *B*, *C* throw a die successively, staking 3*s*, 4*s* 6*d*, 5*s* 6*d*, respectively, the whole to be received by him who first throws an ace: what are they likely to gain or lose by the event?

218. From an observed event, which is known to have sprung from one or other of certain causes, to estimate the probability that any particular one of these exists as the cause of it. [This is said to be the probability, estimated *a posteriori*, of the cause existing.]

Let P_1 denote the probability, *a priori*, (that is, antecedent to the observed event, and wholly irrespective of it,) that the cause in question will exist, p_1 the probability that the event observed will follow, if that cause exist; then P_1p_1 is the probability that the event (i) will happen, and (ii) from that identical cause: and so proceeding similarly for the other causes, we have $P_1p_1 + P_2p_2 +$ &c. or $\Sigma(Pp)$, the probability that the event will come to pass from one or other of the causes, that is, that it will happen in some way.

Again, $\Sigma(Pp)$ being the prob^y that the event will happen, let Q_1 be the prob^y required, viz. the prob^y that, if it happens, it will be from that particular cause existing; then $\Sigma(Pp)Q_1$ is the prob^y that it (i) will happen, and (ii) from that identical cause, which was shewn to be also P_1p_1: hence $Q_1 = \dfrac{P_1p_1}{\Sigma(Pp)}$; so $Q_2 = \dfrac{P_2p_2}{\Sigma(Pp)}$, &c.

Cor. If the causes are all, a priori, *equally* possible, then $P_1 = P_2 =$ &c., and, therefore, $Q_1 = p_1 \div \Sigma(p)$, $Q_2 = p_2 \div \Sigma(p)$, &c.; and if also $\Sigma(p) = 1$, then $Q_1 = p_1$, $Q_2 = p_2$, &c.

Ex. 1. A bag contains 3 balls, and we know only that their colours are (i) white *and* black, (ii) each white *or* black: a ball being drawn is found to be white; what is the probability of drawing again a white ball, the former being replaced?

Here (i) two cases are *equally* possible, *one* white, or *two* white; $\therefore p_1 = \frac{1}{3}$, $p_2 = \frac{2}{3}$, $\Sigma(p) = 1$; and $Q_1 = \frac{1}{3}$, $Q_2 = \frac{2}{3}$; hence the probability of drawing again a white ball is $\frac{1}{3}Q_1 + \frac{2}{3}Q_2 = \frac{5}{9}$:

(ii) three cases are *equally* possible, *one*, *two*, or *three* white; $\therefore p_1 = \frac{1}{3}$, $p_2 = \frac{2}{3}$, $p_3 = 1$, $\Sigma(p) = 2$; and $Q_1 = \frac{1}{6}$, $Q_2 = \frac{1}{3}$, $Q_3 = \frac{1}{2}$; hence prob^y of drawing again a white ball is $\frac{1}{3}Q_1 + \frac{2}{3}Q_2 + Q_3 = \frac{1}{18} + \frac{2}{9} + \frac{1}{2} = \frac{7}{9}$.

Ex. 2. In a bag are three white and two black balls: a person draws one in each hand, and looks at one of them, which proves to be white; what is the probability that the other is (i) white, (ii) black?

Here P_1 = probability that he has drawn two white balls $= \frac{3}{10}$, P_2 = probability that he has drawn one white and one black $= \frac{6}{10}$; $p_1 = 1$, $p_2 = \frac{1}{2}$; $P_1p_1 = \frac{3}{10}$, $P_2p_2 = \frac{3}{10}$; and $Q_1 = Q_2 = \frac{1}{2}$.

Ex. 48.

1. In a bag are four balls, one white, the others either all white or all black: a ball is drawn, and found to be white; what is the chance (i) that the others are white, (ii) that a white ball will again be drawn, the former being replaced.

2. Find the chance in (1), the white ball first drawn being replaced, of drawing (i) a black ball, if one be taken, or, if two be taken, (ii) a black and a white, (iii) a black and *then* a white one?

3. There are three balls in a bag, of whose colour we are ignorant, except that each of them is either white or black: a ball is drawn twice and replaced, and each time it was found to be white: what is the chance of drawing two black in two more trials?

4. In a bag are 7 balls, 2 black, 2 white, and the others doubtful; two balls are drawn and found to be white; what is the chance that the other three are white? and the chance of drawing black and white at the next trial, the two already drawn being removed?

5. Find the chances in (4) when the balls first drawn are found to be (i) both black, (ii) white and black, (iii) white and *then* black.

6. In a bag are three balls, which are known to be all black *or* all white: a white ball is dropped into it, and now a ball is drawn and found to be white: what is the chance that they are *all* white? and the chance, when this is replaced, of again drawing white?

7. Find the chances in (6) if the ball dropped be taken, without being seen, from a bag containing one white and three black balls.

8. Find the chances in [218 Ex. 1], if the number of balls be n.

9. A has 3 sovereigns in one pocket, and 3 shillings in another, he knows not which: he takes a coin from one pocket and transfers it to the other, and then draws out a coin from the latter, which turns out to be a shilling: what is the chance that B, who is to have the contents of the other pocket, will receive two sovereigns? and what is his just expectation?

10. There are two purses, one containing two sovereigns and a shilling, the other two shillings and a sovereign: a person takes a coin from each of them and drops it in the other, the first of which is now given to A, and the second to B: what are their reasonable expectations, (i) as the case is stated, (ii) if A has drawn a coin from his purse and found it to be a sovereign, (iii) if B has done the same?

219. The following are additional Problems of interest in Chances.

Ex. 1. Find the chance of throwing 8 in three throws of a common die, or, which amounts to the same, in one throw of 3 dice.

The No. of ways in which this can be done will plainly be indicated by the coefficient of x^8 in the expansion of

$$(x+x^2+x^3+x^4+x^5+x^6)^3 = x^3(1+x+x^2+x^3+x^4+x^5)^3 = x^3\left(\frac{1-x^6}{1-x}\right)^3,$$

that is, by the coeff. of x^5 in $(1-3x^6+3x^{12}-x^{18})\times(1-x)^{-3}$, or in $(1-x)^{-3}$, (since the terms omitted cannot affect the coeff. of x^5,) and will therefore be 21; but there are $6^3=216$ *possible* throws; hence the chance required is $\frac{21}{216}=\frac{7}{72}$.

Ex. 2. *Concurrent Testimony.* A speaks truth a out of $a+b$ times, and B a' out of $a'+b'$ times: what is the probability of the truth of a fact which they both agree in asserting?

Observe that A's *veracity*, in the sense in which it is here used, will depend, of course, upon his *judgment*, means of knowing the truth, &c. as well as mere *truthfulness*.

Now the fact asserted (i) is true, (ii) is not, (either of which cases let us suppose, for the present, *a priori*, equally probable):

then in (i) they both spoke truth, in (ii) untruth;

hence $p_1 = \dfrac{aa'}{(a+b)(a'+b')} = pp'$ suppose, $p_2 = \dfrac{bb'}{(a+b)(a'+b')} = qq'$;

$$\therefore \text{ (since } P_1 = \tfrac{1}{2} = P_2\text{), we have } Q_1 = \frac{aa'}{aa'+bb'} = \frac{pp'}{pp'+qq'}.$$

So also, if A asserts and B denies, the probability of the truth of the fact may be shewn to be $\dfrac{ab'}{ab'+a'b} = \dfrac{pq'}{pq'+p'q}$, &c.

Of course, if the two suppositions are not, *a priori*, equally probable, we must take into account the different values of P_1, P_2.

It is plain that the same reasoning will apply to any number of such testimonies; and if the veracities of n such witnesses each $=p$, the prob^y of the truth of a fact which they agree in asserting is $\dfrac{p^n}{p^n+q^n}$; or if m assert and n deny it, it is $\dfrac{p^mq^n}{p^mq^n+p^nq^m} = \dfrac{p^{m-n}}{p^{m-n}+q^{m-n}}$, the same as if there were $m-n$ concurring witnesses.

Ex. 3. There is a raffle with 10 tickets, and two prizes value £7 and £3 each: A, B, C, whose veracities are $\frac{4}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, (a, b, c, suppose) respectively, were present at the decision, and report the result to D, who holds one ticket: A and B assert that he has won the £7 prize, C, the £3 prize; what is D's expectation?

Three cases are possible; he may have won £7, £3, or *nothing*, for A, B, C may have all spoken falsely: now for these we have the *a priori* probabilities, $P_1 = \frac{1}{10}$, $P_2 = \frac{1}{10}$, $P_3 = \frac{8}{10}$; also $p_1 = abc' = \frac{2}{3} \times \frac{3}{5} \times \frac{1}{5} = \frac{6}{75}$, $p_2 = a'b'c = \frac{1}{3} \times \frac{2}{5} \times \frac{4}{5} = \frac{8}{75}$, $p_3 = a'b'c' = \frac{1}{3} \times \frac{2}{5} \times \frac{1}{5} = \frac{2}{75}$;
$\therefore P_1p_1 = \frac{6}{750}$, $P_2p_2 = \frac{8}{750}$, $P_3p_3 = \frac{16}{750}$, $Q_1 = \frac{6}{30} = \frac{1}{5}$, $Q_2 = \frac{4}{15}$, $Q_3 = \frac{8}{15}$;
$\therefore$ D's expectation $= \frac{1}{5}$ of £7 + $\frac{4}{15}$ of £3 = £2 4*s*.

Ex. 4. *Traditionary Testimony.* If A states a fact, having received an account of it from B, the fact will be truly stated, if either B has told A the truth and A reported truly, or B has told A untruth and A reported untruly, that is, if they have both spoken truth, or both untruth, the probability of which (using the notation of Ex. 3) is $pp' + qq'$: and, in like manner, the probability of the fact being falsely stated is $pq' + p'q$.

The former of these chances is > the latter, if $p(p' - q') > q(p' - q')$, or $(p - q)(p' - q') > 0$, which requires that, if $p >$ or $< q$, then $p' >$ or $< q'$: so that the probability is in favour of the truth of the fact if A and B are *both* credible persons, or *both* not credible.

If there be n witnesses, of equal veracity p, each of whom has transmitted a statement of a fact to the next, it is easily seen that the chances *for* or *against* the truth of the statement made by the last of them will be the sums of the *odd* or *even* terms respectively of $(p + q)^n$, (bearing in mind that, in a chain of such *traditional* testimony, any *even* number of false statements counteract each other;) that is, they will be $\frac{1}{2}\{(p + q)^n \pm (p - q)^n\}$ respectively.

Thus, if there be five such witnesses, the statement of the last is true, if all *five*, or *three*, or *one* only, speak truth, the probability of which is $p^5 + 10p^3q^2 + 5pq^4$.

Ex. 5. The veracities of A, B, C, D, E are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, (a, b, c, d, e, suppose) respectively: what is the proby of an event, equally likely *a priori* to have happened or not, which A asserts, B denies, having each received it by tradition from C, D, E?

Here the proby of the truth of the statement which reaches A and B, or that arising from the traditional testimony of C, D, E, is $cde + cd'e' + c'de' + c'd'e = \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} + \frac{3}{4} \times \frac{1}{5} \times \frac{1}{6} + \frac{1}{4} \times \frac{4}{5} \times \frac{1}{6} + \frac{1}{4} \times \frac{1}{5} \times \frac{5}{6} = \frac{3}{5}$;

$$\therefore p_1 = \tfrac{3}{5}ab' = \tfrac{1}{10}, \quad p_2 = \tfrac{2}{5}a'b = \tfrac{2}{15}, \quad Q_1 = \tfrac{3}{7}.$$

N.B. It will be supposed in the following Examples, that the fact spoken of is, *a priori*, equally likely to have happened or not, unless it be otherwise mentioned.

Ex. 49.

1. Find the separate chances of throwing 5, 7, 9, or 10, (i) with two, (ii) with three dice.

2. Find the chances of throwing 6, 8, 11, or 16, (i) with three, (ii) with four dice.

3. Shew that, with three dice, there is an equal probability of throwing the number 10 or under, and a number above 10.

4. Compare the chances of throwing, upon two casts with two dice, the number 9 or under, and a number above 9.

5. A speaks truth twice out of three times, and B three times out of four; what is the probability of a fact which (i) they both assert, (ii) A asserts and B denies, (iii) A reports having heard a statement of it from B?

6. Three witnesses, on each of which it is 3 to 2 that he speaks truth, agree in asserting that a certain event occurred: what is the probability that it did so occur? and what, if the probability of their speaking truth be 2 to 1, 3 to 2, and 4 to 3, respectively?

7. Find the chances in (6), if the evidence is traditionary.

8. A speaks truth 2 out of 3 times, and B 3 out of 5 times: what is the probability that they will contradict each other, in the statement of the same fact?

9. A and B being the same as in (8), A brings word to C that he has won a prize of £10: what is C's expectation, if B also brings word (i) to the same effect, (ii) to the contrary? and what, (iii) if A only reported the result, having heard it from B?

10. A and B assert the same event, having each heard a report of it from C; what is the probability of its truth, when the veracities are (i) of each $\frac{3}{5}$, (ii) of A and B, $\frac{3}{4}$, of C, $\frac{2}{3}$, (iii) of A, $\frac{2}{5}$, B, $\frac{2}{3}$, C, $\frac{3}{4}$?

11. Find the last chance in (10), (i) if A, B, C assert the fact independently, (ii) if A asserts, B and C deny it.

12. Find the same, if A assert (i) by tradition from B and C, (ii) having heard the *same* report from each of B and C.

13. In a bag are 2 white and 5 red balls; one of these has been drawn, and A and B each assert that it is a white ball, C denies it, all three having seen the ball: what is the probability that it is a white one, taking the veracities as in the three cases of (10)?

14. A. B, C, D are witnesses of equal character, whose judgment may be relied on twice out of three times: what is the

probability of a fact which (i) they all agree in asserting, (ii) A asserts, having received it by tradition through B, C, D, (iii) A and B assert, having each received it by tradition through C and D, (iv) A asserts, B denies, under the last circumstances?

15. If each can be relied on 4 times out of 5, find the probability of an event, which (i) A, B, C assert, D denies, and (ii) A, B assert, C denies, having each heard from D.

16. A and B being as in (8), what is the probability of a fact which (i) A and B assert, but C denies, who speaks truth 3 out of 4 times, (ii) A asserts, but B and C deny, (iii) A asserts, having heard from B, who has heard from C, (iv) A asserts and B denies, having both heard from C.

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220. The principal application of the doctrine of chances is to the Calculation of Life Insurances and Annuities.

Thus, supposing that, out of 86 persons born, one dies every year till all are extinct, then an Insurance Office would consider that the average duration of life for a *number* of persons, aged 20, would be 66 years, some of them dying earlier, but others later than this. Hence, to insure £100 to be paid on the death of any of these, they would charge such an annual premium, as with its Interest at the end of 66 years to produce to themselves £100, with what profit upon the transaction they may choose to require.

Supposing again a person of that age wished to procure himself an annuity for the remainder of his life. Let R represent £1 with its interest for a year; then the present value of £1 to be paid in one, two, &c. years is $1 \div R$, $1 \div R^2$, &c.; and the chance of his living one, two, &c. years is $\frac{65}{66}$, $\frac{64}{66}$, &c.; therefore the present value of an Annuity of £1 is $\frac{1}{66}\left\{\frac{65}{R} + \frac{64}{R^2} + \&c.\right\}$, which is the amount he must pay down, in order to receive £1 every year he lives, and proportionally for any other Annuity.

The above will give the student a general idea of the mode of calculation in such cases. The practical treatment of the subject will involve many complicated considerations, only worthy of the special attention of those who are actually engaged in them.

MISCELLANEOUS EXAMPLES: PART II.

1. Find the L. C. M. of $2x^3 + (2a - 3b)x^2 - (3a + 2)bx + 3b^2$ and $2x^2 - (3b - 2c)x - 3bc$.

2. Divide $ax^{-1} + a^{-1}x + 2$ by $a^{\frac{1}{3}}x^{-\frac{1}{3}} + a^{-\frac{1}{3}}x^{\frac{1}{3}} - 1$.

3. Shew that $(a - b)^2 + (b - c)^2 + (c - a)^2$
$$= 2\{(a - b)(a - c) + (b - a)(b - c) + (c - a)(c - b)\}.$$

4. A starts from a certain place, and travels a miles the first day, $2a$ the second, $3a$ the third, &c.; after 4 days, B starts to overtake him, travelling $9a$ miles per day. After how many days will he come up with him?

5. Solve in positive integers $4x + 5y + 14z = 49$.

6. If α, β, are the roots of $x^2 + px + q = 0$, find the value of $\alpha^2 + \alpha\beta + \beta^2$, $\alpha^3 + \beta^3$, and $\alpha^4 + \alpha^2\beta^2 + \beta^4$.

7. Divide a given number a into n parts in the ratio of 1, 2, 3, &c.

8. The No. of Variations of $m + n$ things, two together, is 56, and of $m - n$ things is 12: find the No. of Combinations of m things, n together.

9. Solve the equation $\sqrt{(a^2 + cx)} + \sqrt{(a^2 - cx)} = \sqrt{(2acx)}$.

10. £P is left among A, B, C, so that, at the end of a, b, c, years respectively, when they come to age, they will possess equal sums: find the present share of each at Comp. Interest.

11. Reduce $\dfrac{y^3 - (2a + b)y^2 + (2ab + a^2)y - a^2b}{3y^2 - (4a + 2b)y + 2ab + a^2}$.

12. If $2m = x + x^{-1}$ and $2n = y + y^{-1}$, express $mn + \sqrt{\{(m^2 - 1)(n^2 - 1)\}}$ in terms of x and y.

13. Find the n^{th} term of an A. P., when the sum of $n + 1$ terms is $(n + 1)(n + 1\frac{1}{3})$.

14. A lb. of tea and 3 lbs. of sugar cost together 6*s*; but, if sugar were to rise 50 *per cent* and tea 10 *per cent*, they would cost 7*s*: find their prices.

15. If $a_1 : a_2 :: a_2 : a_3 ::$ &c. $:: a_{n-1} : a_n$, then $a_1 : a_2 :: \sqrt[n-1]{a_1} : \sqrt[n-1]{a_n}$.

16. A, counting a box of oranges, which was known to contain under 200, observed that when he told them by 2, 3, 4, 5, 6 at a time he had none over, but when by 7, he had 5 over: how many had he?

17. Two vessels contain each a mixture of wine and water. In A the wine : water :: 1 : 3, in B :: 3 : 5; how much must be taken from each to make 5 gals. of wine and 9 of water?

18. Obtain the square root of $1+(1-c^2)^{-\frac{1}{2}}$ in the form $\alpha^{\frac{1}{4}}+\beta^{\frac{1}{4}}$.

19. Solve the equation $\dfrac{x}{a+x}+\dfrac{a}{\sqrt{(a+x)}}=\dfrac{b}{x}$.

20. A flag-staff (c) stands at the top of a tower, whose height is a. Find the distance from the foot of the tower, at which the flag-staff subtends the greatest angle.

21. Decompose $1-\left\{\dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right\}^2$ into simple factors.

22. Form the equation whose roots are $\sqrt{m}\div\{\sqrt{m}\pm\sqrt{(m-n)}\}$.

23. If $a:b::c:d$, then $a+b:c+d::a^2(c-d):c^2(a-b)$.

24. Shew that in the series 1, 3, 5, &c., the first half of any even No. of terms has to the second half a fixed ratio.

25. If A had travelled half-a-mile an hour faster, he would have finished his journey in $\frac{4}{5}$ of the time: whereas, if he had travelled half-a-mile an hour slower, he would have been $2\frac{1}{2}$ hrs longer on the road. How many miles did he travel?

26. Find the No. of Combinations that can be made of the letters in the word *Notation*, taken 3 together.

27. A sum of £8 6*s* 6*d* is made up of sovereigns, shillings, and sixpences; find the No. o. coins of each kind, it being known that the amount of the shillings is a guinea less than that of the sovereigns, and a guinea and a half more than that of the sixpences.

28. Find the equated time of payment, at 5 per cent Simp. Int., for sums of £400 and £2100, due at the end of 2 years and 8 years respectively.

29. Solve the equations $(x+y)(x^3+y^3)=76$, $(x+y)^3=64(x-y)$.

30. In a certain country, the births in a year amount to an m^{th} of the whole population, and the deaths to an n^{th}: in how many years will the population be doubled?

31. Express in the form of the sum of two simple surds the roots of the equations,

(i) $x^4-2ax^2+b^2=0$, (ii) $4x^4-4(1+n^2)a^2x^2+n^2a^4=0$.

32. Find the relation between p, q, r, s, (i) when px^3+qx^2+rx+s is a perfect *cube*, (ii) when $x^4+px^3+qx^2+rx+s$ is a square.

33. A train, an hour after starting, meets with an accident which detains it an hour, after which it proceeds at $\frac{3}{5}$ of its former rate, and arrives 3 hrs behind time: but had the accident happened 50 miles farther on the line, it would have arrived $1\frac{1}{3}$ hr sooner. Find the length of the line.

34. If $x + 2\left\{x - \dfrac{1}{n-1}\right\} + 3\left\{x - \dfrac{2}{n-1}\right\} + 4\left\{x - \dfrac{3}{n-1}\right\} + \&c.$ to n terms $= 0$, shew that $x = \frac{2}{3}$.

35. Find the No. which when divided by 7, 8, 9, leaves rem[rs] 1, 2, 3, and such that the sum of the three quotients is 570.

36. Find the coefficient of x^n in $\log_e \dfrac{1}{1 - ax + x^2}$.

37. Given the No. of Var[ns] of $2n + 1$ things, $n - 1$ together : No. of Var[ns] of $2n - 1$ things, n together, :: 3 : 5; determine n.

38. Given $\log 1\frac{1}{4} = .0969100$, and $\log .\dot{1} = \bar{1}.0457575$, find the logs of $2\frac{1}{2}$, $2\frac{1}{4}$, $.\dot{2}$, $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{3}}$, and $\frac{3}{40}\sqrt[5]{(.0027)^3} \div \sqrt[3]{(.015)^5}$.

39. Solve the equations $\sqrt{(ax)} + \sqrt{(by)} = \frac{1}{2}(x + y) = a + b$.

40. What is the chance of throwing (i) 10, (ii) 20, in three throws with two dice?

41. Shew that $N(N^4 - 1)$ or $N(N^2 + 20)$ will be divisible by 48, according as N is an *odd* or *even* number.

42. Find the value of $\sqrt{\left(\dfrac{1}{x^2 - x} + \dfrac{1}{4x^2}\right)} - \dfrac{1}{2x}$, when $x = 0$.

43. A person distributed p shillings among n persons, giving $9d$ to some and $15d$ to the rest. How many were there of each?

44. Obtain the square root of

$$(\sqrt[x]{\sqrt[y]{a^{-1}}})^2 + \sqrt[x]{(b\sqrt[y]{a})} + 2\sqrt{(\sqrt[z]{b})}.\sqrt{\sqrt[x]{\sqrt[y]{\sqrt[z]{a^{x-2z}}}}}.$$

45. Divide 75 into two parts so that, when divided by 5 and 6 respectively, they may each give the same remainder 4.

46. Write down the general terms of $(a^2 - x^2)^{\frac{2}{3}}$ and $(a^2 + x^2)^{-\frac{2}{3}}$.

47. If $\dfrac{a}{b} = \dfrac{c}{d}$, shew that $\dfrac{a^2 + b^2}{c^2 + d^2} = \dfrac{ab}{cd} = \left(\dfrac{a+b}{c+d}\right)^2 = \dfrac{ma^2 - nab + pb^2}{mc^2 - ncd + pd^2}$.

48. A country trebles its population in a century: what is the increase in one year per million, given $\log 3 = .4771213$, $\log 101\frac{1}{10} = 2.0047512$, $\log 101\frac{11}{100} = 2.0047941$.

49. Solve (i) $\left(\dfrac{a+x}{a-x}\right)^2 = 1 + \dfrac{cx}{ab}$, (ii) $\dfrac{a^2x}{y^2z^2} = \dfrac{b^2y}{x^2z^2} = \dfrac{c^2z}{x^2y^2} = 1$.

50. Compare the chances of throwing 4 with one die, 8 with two, and 12 with three dice, having two throws in each case.

51. If $(a^2+bc)^2 . (b^2+ac)^2 . (c^2+ab)^2 = (a^2-bc)^2 . (b^2-ac)^2 . (c^2-ab)^2$, shew that either $a^3+b^3+c^3+abc=0$, or $a^{-3}+b^{-3}+c^{-3}+a^{-1}b^{-1}c^{-1}=0$.

52. If N, N' be two consecutive numbers, neither of them a multiple of 3, shew that $N^3+N'^3$ is a multiple of 9 and $(N^3 \sim N'^3)-7$ of 54.

53. Evaluate $\dfrac{x^3-1}{x^3-2x^2+2x-1}$ and $\dfrac{x^2(y+1)-xy-1}{x^2(y-1)-x(y-2)-1}$, when $x=1$.

54. Given $\log \frac{1}{2} = \bar{1}.6989700$, $\log \frac{1}{3} = \bar{1}.5228787$, find the logs of $\sqrt{3}$, $\sqrt[3]{2}$, $\sqrt[4]{6}$, $\sqrt[5]{(1.44)^2}$, $\frac{2}{3}\sqrt[3]{.05}$, $\frac{12}{25}\sqrt[4]{(1.6)^3} \times \sqrt[3]{(21.6)^4}$.

55. Find the n° of terms in $(a+b^2+c^3)^{12}$ and the coeff. of $a^8b^4c^6$.

56. Expand $(1+x)^{\frac{3}{2}}$ to 5 terms by reversion of series.

57. The sum of n terms of an A. P. is $pn+qn^2$: find the m^{th} term.

58. A gives B a bill for £a, due at the end of m years, in discharge of a bill for £b, due at the end of n years: for what sum should B give A a bill due at the end of p years, to balance the account at Comp. Int.?

59. Solve the equation $\left(\dfrac{x}{x-1}\right)^2 + \left(\dfrac{x}{x+1}\right)^2 = n(n-1)$.

60. In a bag are five red balls and one white: find the chance that in three drawings (replacing the ball drawn after each) there will have been drawn (i) three white balls, (ii) three red balls, (iii) two red and one white.

61. Find the G. C. M. of

$$(ax+by)^2-(a-b)(x+z)(ax+by)+(a-b)^2xz$$

and $(ax-by)^2-(a+b)(x+z)(ax-by)+(a+b)^2xz$.

62. Evaluate $\left(\dfrac{x^3-a^3}{x^4-a^4}\right)^2$ and $\dfrac{1-e^{1-\frac{x}{a}}}{x^4-a^4}$, when $x=a$.

63. Find the *maximum* or *minimum* value of $\dfrac{4m^2x^2+1}{(4m^2+1)x}$.

64. A person bought for £100 a hundred head of cattle, for which he paid 30*s*, 10*s*, and £10 a head respectively: how many did he buy of each, it being known that the sheep and oxen were together under 20, the rest being pigs?

65. Determine m and n in terms of a and b, so that $\dfrac{ma+nb}{m+n}$ may be the Arithmetic mean between m and n, and the Geometric mean between a and b.

66. Find the coefficients of $x^{\frac{3}{2}}$ and x^2 in $(a - bx^{\frac{1}{2}} + cx - dx^{\frac{1}{4}})^7$.

67. The mail from A starts for B at p hrs P.M., and that from B for A at q hrs A.M. Now, if the mail from B, n hrs after starting, meet a mail from A, and again at a miles from A meet another mail from A, what is the distance from A to B?

68. If $x^2 = ay - y^2 + a^{-1}y^3 -$ &c., express y in terms of x.

69. Solve $\sqrt{(x^2 + 2bx + a^2)} - \sqrt{(x^2 - 2bx + a^2)} = 2\sqrt{(x^2 - b^2)}$.

70. Find the last two chances in [60], if the balls are *not* replaced.

71. Simplify $(x+1)(x^2+x+1)^{-1}+(x-1)(x^2-x+1)^{-1}+2(x^4+x^2+1)^{-1}$.

72. Shew that $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{(2+\sqrt{3})}} + \dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{(2-\sqrt{3})}} = \sqrt{2}$.

73. Shew that $a : b :: a^2 + ac + c^2 : b^2 + bc + c^2 :: (a+c)^2 : (b+c)^2$, if c be a mean proportional between a and b.

74. Write the general terms of $(1+x)^{-3}$, $(1-x)^{-\frac{1}{3}}$, and $(a^2 - ax)^{\frac{3}{4}}$.

75. Find what values of x make $\dfrac{x^3 - 2ax^2 - a^2x + 2a^3}{x^3 - ax^2 - 4a^2x + 4a^3}$ a vanishing fraction, and evaluate it in those cases.

76. Given S the sum, and s^2 the sum of the squares, of the terms of an infinite G.P., shew that its sum to n terms

$$= S\left\{1 - \left(\frac{S^2 - s^2}{S^2 + s^2}\right)^n\right\}.$$

77. A and A' can separately produce effects a and a' in times t and t': in what time could they together produce an effect c? Shew that if $c = a + a'$, this time will be the A. or G. mean between t and t', according as $t : t' = a : a'$ or $a^2 : a'^2$.

78. Shew that the n° of different Combinations of n things taken 1, 2, 3, &c. n together, of which p are of one sort, q of another, r of another, &c., is $(p+1)(q+1)(r+1)\ldots - 1$.

79. Solve the equations $ax + \dfrac{b^3}{y} = a^2 = (a^2 - bx)\dfrac{y}{a}$.

80. A had in his pocket a sovereign and four shillings; taking out two coins at random, he promises to give them to B and C: what is the worth of C's expectation?

81. Shew that $(a+b-c)^3 + (b+c-a)^3 + (c+a-b)^3 > 3abc$.

82. Eliminate x and y from $ax+by=c\sqrt{(x^2+y^2)}$, $a'x+b'y=c'\sqrt{(x^2+y^2)}$.

83. Simplify $\dfrac{7+\sqrt{-3}}{2-\sqrt{-3}} + \dfrac{8+3\sqrt{-3}}{2+\sqrt{-3}} - \dfrac{4(2-\sqrt{-3})}{1-\sqrt{-3}}$.

84. Resolve $\dfrac{x^2+x+1}{(x-1)(x-2)(x-3)}$ and $\dfrac{x^2-x+1}{x^2(x-1)^2}$ into partial fractions.

85. If a, b, c are in H.P., then $a : a-b :: a+c : a-c$, and $a^2+c^2>2b^2$.

86. The sides of a square are bisected and joined, the side of the square thus formed being a foot less than that of the former; the sides of the second square are bisected and joined, and so on for ever: find the sums of the perimeters and areas of all the squares.

87. If $y^3 - axy - b^3 = 0$, expand y in terms of x.

88. When wax candles are 2*s* 6*d* a lb, a composition is invented of such a nature, that a candle made of it will burn two-thirds of the time, in which a wax candle of the same thickness and one-fourth as heavy again will continue burning. If the two candles give an equally bright light, what must be charged per lb. for the composition, that it may be as *cheap* as wax?

89. Solve the equation $\frac{1}{a}\sqrt{(a+x)} + \frac{1}{x}\sqrt{(a+x)} = \frac{1}{b}\sqrt{x}$.

90. Find the worth of *C*'s expectation in [80], (i) if it is seen that one of the two coins which *A* has drawn is a shilling, (ii) if *B*, having received his coin, finds it to be a shilling, (iii) if *both* these suppositions are made together.

91. Find the n° of div^rs of 2160, and the n° of Nos. less than it, and prime to it.

92. Shew that $7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80} = \log 2$.

93. Expand by the Bin. Theor. $(1 - ax + bx^3)^{-\frac{1}{2}}$ to five terms.

94. Shew that $\sqrt[3]{N} = a\,\frac{2N + a^3}{N + 2a^3}$ nearly, a being the integer next greater than $\sqrt[3]{N}$.

95. In every G.P. of an odd number of terms, the sum of the squares of the terms = product of sum of all the terms by excess of the odd terms above the even.

96. If the G. mean between x and y : the H. mean :: $m : n$, then $x : y :: m + \sqrt{(m^2 - n^2)} : m - \sqrt{(m^2 - n^2)}$.

97. Find the sum of all the numbers of n places which can be made with n digits p_0, p_1, &c. in the scale of r. What is the greatest possible value of that sum for the given radix?

98. The weight of a spherical shell is $\frac{7}{8}$ of what it would have been if wholly solid, Given that the weight of a sphere $\propto$ (diam)^3, compare the inner and outer radii: and, if the inner be increased by one-half, find in what ratio the present weight will be reduced.

99. Solve the equation

$$\sqrt{(x+\sqrt{x})} - \sqrt{(x-\sqrt{x})} = a\sqrt{\frac{x}{x+\sqrt{x}}}.$$

100. The veracities of A, B, C, are as $\tfrac{2}{3}$, $\tfrac{3}{4}$, $\tfrac{4}{5}$: a shilling is tossed, and A and B assert that it has fallen *head*, C that it has fallen *tail:* what is the chance that it fell *head?*

101. If $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$ have a common root, prove that $(ac'-a'c)^2+(ab'-a'b)(cb'-bc')=0$. What is the condition that they may have both roots common?

102. Shew that the sum of any two consecutive triangular numbers is a square.

103. If a, b, c, d, &c. are in G. P., find the sum of n terms of the series $(a^3+b^3)^{-1}+(b^3+c^3)^{-1}+(c^3+d^3)^{-1}+$&c. in terms of a and b.

104. If $2a = 3b$, find the numerical values of

$$\frac{a-b}{a+b},\quad \frac{a^2-b^2}{a^2+b^2},\quad \frac{a^3+3b^3}{2a^3-\frac{8}{9}b^3},\quad \frac{5a^2b^2}{2a^4+3b^4},\quad \frac{ab(a+b)}{(2a+3b)(3a-2b)^2}.$$

105. Evaluate $\{(1-x)^{-\frac{3}{2}}-(1+x)^{\frac{3}{4}}\} \div (x^3-x^4)^{\frac{2}{3}}$, when $x=0$.

106. If m shillings in a row reach as far as n sovereigns, and a pile of p shillings be as high as a pile of q sovereigns, compare the values of equal bulks of gold and silver.

107. S_1, S_2, &c. are the sums of m AR. series, each to n terms, the first terms being 1, 2, 3, &c. and the differences 1, 3, 5, &c.: shew that S_1+S_2+ &c. $=\tfrac{1}{2}mn(mn+1)$.

108. If $a \div b$ be an irreducible fraction, and b any number prime to 3, shew that, when the fraction is converted to a decimal, the period will be divisible by 9, and the sum of the remainders will be a multiple of b.

109. (i) $\dfrac{a+x+\sqrt{(a^2-x^2)}}{a+x-\sqrt{(a^2-x^2)}}=\dfrac{b}{x}$; (ii) $\dfrac{x^2-a^2}{cx}+\dfrac{cx}{x^2-a^2}=b$.

110. The veracities of A, B, C, being $\tfrac{2}{3}$, $\tfrac{3}{4}$, $\tfrac{4}{5}$, C asserts that I have won a prize of £10 in a certain raffle, where there are 10 tickets, of which I hold *two:* what is the value of my expectation, if I find that C has only heard a report of the case from B, who again had only heard from A?

111. Given log 2 = .3010300, find those of 5, .016, $\tfrac{1}{2}$, 6.25, $1\tfrac{3}{5}$, $15\tfrac{5}{8}$.

112. Shew that $x+\dfrac{m}{nx}>1+\dfrac{m}{n}$, unless x lies between 1 and $\dfrac{m}{n}$.

113. Write down the general terms of

$$(a^2-5ax)^{-\frac{3}{5}},\quad (a^3+3a^2x)^{\frac{5}{3}},\ \text{and}\ (ax+x^2)^{-4}.$$

114. Separate $\dfrac{x^3+a^3}{x^2(x-a)^2}$ and $\dfrac{a^2x^2}{x^3+a^3}$ into partial fractions.

115. Find the coeff. of x^{13} in the expansion of $(ax-bx^3+cx^5-\&c.)^6$.

116. If N and n be nearly equal,

$$\text{then } \sqrt{\frac{N}{n}} = \frac{N}{N+n} + \frac{1}{4} \cdot \frac{N+n}{n} \text{ very nearly.}$$

117. If in [116] $\dfrac{N}{N+n}$ and $\dfrac{1}{4} \cdot \dfrac{N+n}{n}$ have their first p decimal places the same, shew that the approximation may be relied on to $2p$ decimals at least; and hence find $\sqrt{30}$ to eight decimal places.

118. If the m^{th} term of an A.P. be n, and the n^{th} term m, how many terms must be taken so as to give the sum $\frac{1}{2}(m+n)(m+n-1)$? and what will be the last of them?

119. Solve the equations

$$\left.\begin{aligned} &1 + \tfrac{1}{5}x^2y^{-2} + \tfrac{1}{25}x^4y^{-4} + \&c. \textit{ ad inf.} = 5x^2y^{-2} \\ &1 + \tfrac{1}{2}(x+y) + \frac{1.3}{2.4}(x+y)^2 + \frac{1.3.5}{2.4.6}(x+y)^3 + \&c. \textit{ ad inf.} = \sqrt{1.25} \end{aligned}\right\}$$

120. There are 7 balls in a bag, one of them a white one. A and B stake each 3*s* 6*d*, the whole to be won by whichever shall first draw the white ball, the balls not being replaced when drawn. A has the first draw: what are their expectations? and what should B have staked, that A's drawing first might give him no advantage?

121. Find the n° of divisors of 140, and the n° of numbers less than 140 and prime to it: express generally the rational values of x and y which satisfy the equation $140x = y^3$, and find how many integral solutions there are of $xy = 10^{2n}$.

122. Given that $e^x = y + \sqrt{(1+y^2)}$, shew that $y = \frac{1}{2}(e^x - e^{-x})$; and prove that $\sqrt{\{c + \sqrt{(2ac-a^2)}\}} + \sqrt{\{c - \sqrt{(2ac-a^2)}\}} = \sqrt{(2a)}$.

123. Find a series of fractions converging to $\frac{351}{965}$ and $\sqrt{28}$.

124. If $c = a - b$, and is very small compared with a and b, then

$$a^2b^2(a^2 - a^2x^2 + b^2x^2)^{-\frac{3}{2}} = a - 2c + 3cx^2 \text{ nearly.}$$

125. Resolve $\dfrac{x+a}{x(x-a)^2}$, $\dfrac{x^3+2ax+3a^2}{(x^2-a^2)^2}$, $\dfrac{x^2-ax+a^2}{x(x^3-a^3)}$ into partial fractions.

126. If α, β, γ, be the roots of $x^3-2x^2+3x-4=0$, find the values of

$$\alpha^{-1} + \beta^{-1} + \gamma^{-1},\ \ \alpha^2 + \beta^2 + \gamma^2,\ \ \alpha^{-2} + \beta^{-2} + \gamma^{-2}.$$

127. The value of diamonds $\propto$ (weight)^2, and the (value)^2 of rubies $\propto$ (weight)^3. A diamond of a carats is worth m times a ruby of b carats, and both together are worth £c. Find the values of a diamond and ruby, each of x carats.

128. A vessel (A) contains a gallons of wine, another (B) contains b gallons of water: c gallons are taken from each and poured into the other, and this operation is continually repeated. Shew that, if $c = ab \div (a + b)$, the quantity of wine in each vessel will remain always the same after the first operation.

129. (i) $\left(\dfrac{x+a}{x+b}\right)^{\frac{1}{2}} - \dfrac{a-b}{2(x+c)} = 1$: (ii) $c^2x^4 - 2c^2x^3 + 2x - 1 = 0$.

130. A bets B 3s to 1s that, with two dice at one cast, he will throw *seven* before B throws *four*, each having a pair of dice and throwing together: what is the value of B's expectation, and what odds ought A to have given?

131. Prove that, if $x = \sqrt{(a^2 + s^2)}$ and $a\,\epsilon^{\frac{y}{a}} = x + \sqrt{(x^2 - a^2)}$, then

$$2x = a\left(e^{\frac{y}{a}} + e^{-\frac{y}{a}}\right),\ \ 2s = a\left(e^{\frac{y}{a}} - e^{-\frac{y}{a}}\right).$$

132. Represent $\sqrt{(2n\sqrt{-1})}$ in the form of a binomial surd; and shew also that $\sqrt{(4 + 3\sqrt{-20})} + \sqrt{(4 - 3\sqrt{-20})} = 6$.

133. Given $\log 1\frac{2}{5} = .1461280$, $\log .144 = \bar{1}.1583625$, and $\log .0441 = \bar{2}.6444386$, find the logs of the nine digits.

134. Eliminate x and y from each of these sets of equations;

$$\text{(i)}\ \ x + y = z,\ \ x^2 + y^2 = a^2,\ \ x^3 + y^3 = b^3,$$
$$\text{(ii)}\ \ x + y = a,\ \ xy = z^2,\ \ x^7 + y^7 = b^7.$$

135. Find the convergents to $\frac{421}{972}$ and $\frac{1096}{9119}$, and the first four to the value of $\sqrt{11}$.

136. If a be the first of n GEOM. means between a and b, then $a : b :: a^{n+1} : a^{n+1}$.

137. In [128] shew that generally the quantity of wine in B after r operations will be $\dfrac{ab}{a+b}(1 - p^r)$, where $c = \dfrac{ab}{a+b}(1 - p)$.

138. Find the number of men in a hollow equilateral wedge, the ranks being r deep, and the outer one containing n men.

139. Solve the equations $\dfrac{x^2}{y^2} + \dfrac{2x+y}{\sqrt{y}} = 20 - \dfrac{y^2+x}{y}$, $x + 8 = 4y$.

140. In [130] determine B's expectation, and the odds A should have given, (i) if A, and (ii) if B, has the *first* throw.

141. If $c = a\sqrt{(1 - b^2)} + b\sqrt{(1 - a^2)}$, then

$$(a + b + c)(a + b - c)(a - b + c)(b + c - a) = 4a^2b^2c^2.$$

142. Sum the series $1 - 2n + 3\dfrac{n(n+1)}{1.2} - 4\dfrac{n(n+1)(n+2)}{1.2.3} + \&c.$; and, if $n < 1$, shew that the terms continually decrease to the r^{th}, so long as $rn < 1$, and after that increase.

143. Find the first four convergents to 3.14159, and also to the ratio of 5 h. 48 min. 51 sec. to 24 h.

144. If the equation $x^2 + px + q = 0$ have equal roots, shew that $ax^2 + p(a+b)x + q(a+2b) = 0$ has one of them, and find the other.

145. Divide unity into four parts in A. P., so that the sum of their cubes may be $\frac{1}{10}$.

146. Shew that $(a_1b_1 + a_2b_2 + \&c.)^2 < (a_1^2 + a_2^2 + \&c.)(b_1^2 + b_2^2 + \&c.)$, unless $a_1 : b_1 = a_2 : b_2 = \&c.$

147. Find the coefficient of x^n in $(a + bx + cx^2)\,e^{-x}$.

148. Given A, my income, a the premium for assuring £100, r the rate of Int. per cent per annum: find what sum I must lay out in insuring my life, so that my executors may receive a sum, whose Int. shall equal my reduced income.

149. (i) $nx = \{\sqrt{(1+x)} - 1\}\{\sqrt{(1-x)} + 1\}$.
(ii) $x^3 = ax + by,\ y^3 = bx + ay$.

150. There are two bags, in one of which are one white ball and two black, in the other three white and one black. Find the chance of a person drawing (i) a white ball, (ii) white and black in two trials, balls replaced, (iii) three white in three trials, balls not replaced.

151. Divide an odd number $2n + 1$ into two integers, so that their product may be the greatest possible.

152. Find the value of $1 - n^2 + \left\{\dfrac{n(n-1)}{1.2}\right\}^2 - \left\{\dfrac{n(n-1)(n-2)}{1.2.3}\right\}^2 + \&c.$

153. From $\log\sqrt{\frac{1}{200}} = \bar{2}.8494850$ and $\log\sqrt[3]{\frac{1}{300}} = \bar{1}.1742929$, find the value of $\sqrt[5]{[\sqrt{2}\times\{\sqrt{3}\div(.0032)^{\frac{4}{3}}\}\div\{\sqrt[3]{1.5}\times\sqrt{.25}\}\times(1800)^{\frac{3}{4}}]}$, given the *mant.* for 19048, 1904825 to be 2798494, 2798551.

154. In any G. P. the sum of any two terms is > sum of any two between them, equally distant from the extremes.

155. Determine which is the greatest term of $(3 + 5x)^8$, when $x = \frac{1}{2}$, and the greatest coefficient of $(1 + x)^{\frac{11}{\sqrt{2}}}$.

156. Shew that the product of $n + 1$ quantities, each of which is the sum of two squares, can be expressed as the sum of two squares in 2^n different ways.

157. There is a number of series in A.P., whose common differences are 1, 2, 3, &c.: shew that, if the sum of n terms of each of these be n^2, their first terms will form a decreasing A.P., whose first term is $\frac{1}{2}(n+1)$ and common diff. $\frac{1}{2}(n-1)$.

158. If n be very great, shew that

$$a^r + (a+b)^r + (a+2b)^r + \&c. \text{ to } n \text{ terms} = \frac{n^{r+1}b^r}{r+1}.$$

159. (i) $x^m y^n = a^m b^n c^m$, $x^n y^m = a^m b^n c^n$. (ii) $\left(\frac{x-a}{x-b}\right)^3 = \frac{x-2a+b}{x+a-2b}$.

160. If in [150] the first drawn ball be *white*, find the chance that it has been drawn out of the first bag. What is it, if also the second be *black*, the former not having been replaced?

161. The diff. between any No. and that No. inverted is div. by 9.

162. Find the sum of the cubes of the roots of $x^2 - x + 1 = 0$.

163. If $n = a^2 + b$, where b is very small, $\sqrt{n} = a + \frac{2ab}{4a^2+b}$ nearly.

164. Evaluate $\frac{a}{x-a}\sqrt[3]{\log_e \frac{x}{a}}$, and $\frac{(x^2-a^2)^{\frac{3}{2}} + (x-a)}{(1+x-a)^3 - 1}$, when $x = a$.

165. Write down the general terms of

$$(a^2+x^2)^{-2},\ (a^2-x^2)^{-\frac{1}{5}},\ \text{and } (a^2-x^2)^{\frac{3}{2}}.$$

166. Prove that 12321, 1234321, 123454321, &c., are squares whatever be the scale of notation.

167. S is the sum of three terms in G.P., the first being 1, S' of their reciprocals: shew that the sum *ad inf.* $= S \div \left\{1 - \left(\frac{S}{S'}\right)^{\frac{3}{2}}\right\}$.

168. If p_r denote the coefficient of x^r in $(a_0 + a_1 x + a_2 x^2 + \&c.)^{\frac{1}{2}}$, shew that $p_0 p_r + p_1 p_{r-1} + p_2 p_{r-2} + \&c. + p_r p_0 = a_r$.

169. Solve $\sqrt{(x-y)} + \frac{1}{2}\sqrt{(x+y)} = \frac{x-1}{\sqrt{(x-y)}}$, $x^2 + y^2 = \frac{34}{15}xy$.

170. The skill of A is double of that of B: find the odds against A's winning four games before B wins two.

171. Given log 2 = .3010300, log 3 = .4771213, find the logs of 3.2, $1\frac{1}{2}$, $\frac{2}{3}$, 15, .0054, $14\frac{2}{5}$, 1.8, 8.1.

172. If $a : b = c : d = e : f$, prove that $(a^2 + c^2) f^2 = (b^2 + d^2) e^2$, and

$$a^{\frac{3}{2}} - \sqrt{(ace)} + e^{\frac{3}{2}} : (\sqrt{a} + \sqrt{c} + \sqrt{e})^3 :: b^{\frac{3}{2}} - \sqrt{(bdf)} + f^{\frac{3}{2}} : (\sqrt{b} + \sqrt{d} + \sqrt{f})^3.$$

173. If n A., G., or H. means be inserted between a and c, obtain the m^{th} mean in each case.

174. Ev. $\dfrac{x-a+\sqrt{(2ax-2a^2)}}{\sqrt{(x^2-a^2)}}$ and $\dfrac{x\sqrt{(3a^2x-2x^4)}-ax\sqrt[5]{(a^4x)}}{a-\sqrt[4]{(ax^3)}}$, when $x=a$.

175. Obtain the general terms of

$$(a^3-x^3)^{-\frac{1}{4}},\ (a^2-ax)^{\frac{1}{3}},\ \text{and } (a+5x)^{\frac{2}{5}}.$$

176. If $A \propto B$, $B^2 \propto AC$, and $C \propto \sqrt[3]{A^2D}+\sqrt[3]{AB^2}$, shew that $m\sqrt[4]{AD}-n\sqrt[4]{BC} \propto p\sqrt{A}+q\sqrt{D}$.

177. A watch stood at 11h. 59m. 49s., when it was Noon by a clock; and two mornings after, when it was 9h. by the clock, the watch was at 8h. 59m. 58s. The clock being known to gain $\frac{4}{5}$ sec. in 24h., find the gaining rate of the watch.

178. A bequeaths to his eldest child an n^{th} of his property + £P, to the second an n^{th} of the remainder + £$2P$, and so on. In the end they all share alike: how many were they, and what sum did each receive?

179. Solve $x^3+y^3+xy(x+y)=13$, $(x^2+y^2)\,x^2y^2=468$.

180. Shew that it is probable that an ace will be thrown at least once in four throws with a single die: and determine the number of times it must be thrown, so that it may be a probability of $p:1$ that an ace will be thrown at least *once*.

181. Shew that any common number N is divisible by 7, when $p_0+3p_1+9p_2+$ &c. is so divisible, where p_0, p_1, &c. are the digits, reckoning from the end of the number.

182. The present value of an annuity A to continue for n years is a, and for $2n$ years is a': find n and the rate of Interest.

183. Find five nos in A.P., whose sum shall be 25 and product 2520.

184. Write down the general terms of $(a^{\frac{1}{3}}-x^{-\frac{1}{3}})^{-6}$ and $(c+c^{\frac{2}{3}}x^{\frac{1}{3}})^{-1}$.

185. Resolve into partial fractions

$$\frac{x^3+x^2+2}{x(x+1)^2(x-1)^2} \text{ and } \frac{3x^2-2x+1}{x^6-1}.$$

186. Reduce $a\sqrt[4]{-1}+b\sqrt[3]{-1}$ to the form $\alpha+\beta\sqrt{-1}$.

187. Find the sum of

$$pa^m \pm m(p+q)a^{m-1}b+\tfrac{1}{2}m(m-1)(p+2q)a^{m-2}b^2 \pm \text{\&c.}$$

188. Between each of $m+1$ pairs of quantities, (x, y), $(x, 2y)$, $(x, 4y)$, &c. are inserted m GEOM. means, and M_1, M_2, M_3, &c. are the m^{th} means respectively: prove that

$$\frac{M_1}{M_2}+\frac{M_2}{M_3}+\text{\&c.}=\sqrt[m+1]{\frac{m^{m+1}}{2^m}}.$$

189. Solve the equation $\sqrt{\dfrac{\sqrt[n]{a}-\sqrt[n]{x}}{\sqrt[n]{x^2}}}-\sqrt{\dfrac{\sqrt[n]{a}-\sqrt[n]{x}}{\sqrt[n]{a^3}}}=\sqrt[2n]{\dfrac{x}{b^2}}$.

190. A and B play at a game, in which their skills are as $3:5$; find the chance of A's winning at least three games out of four.

191. If $m>3$, then $\sqrt[3]{m}>\sqrt[4]{(m+1)}$: also $\sqrt{(-\tfrac{1}{12}+\sqrt{-\tfrac{1}{3}})}+\sqrt{(-\tfrac{1}{12}-\sqrt{-\tfrac{1}{3}})}=1$.

192. If $a_1 : a_2 :: a_2 : a_3 :: a_3 : a_4$ &c., then

$$(a_1^2+a_2^2+\&c.)\,(a_2^2+a_3^2+\&c.)=(a_1a_2+a_2a_3+\&c.)^2.$$

193. Shew that $\{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{2}{3}}\}\div\{(1+x)+\sqrt{(1+x)}\}=1-\frac{5}{6}x$, nearly, when x is *small*: and find its approximate value when x is *large*.

194. Given $\log\frac{1}{2}=\bar{1}.69897$, find x from the equation $20^x=100$.

195. Expand $\dfrac{x-bx^3+dx^5}{1-ax^2+cx^4}$ to five terms by Ind. Coefficients.

196. Find the limits between which lie all real values of x and y, which satisfy the equation $(n^2+1)\,(x^2+y^2-a^2)=4nxy$.

197. If S_n denote the sum of n terms of an A. P., shew that $S_n+S_{n+1}+$ &c. to n terms $=\frac{1}{2}n\,(3n-1)\,a+\frac{1}{6}n\,(n-1)\,(7n-2)\,b$.

198. If the prices of p and q cubic feet of timber be £a and £b, find the price of a tree containing r cubic feet, the values of timber and bark being proportional to the m^{th} and n^{th} powers of the quantity of timber in the tree.

199. Solve the equations $x+y=5$, $(x^2+y^2)\,(x^3+y^3)=455$.

200. A, B, C, D, in order cut a pack of cards, replacing them after each cut, on condition that the first who cuts a heart shall win. What are their respective chances of success?

201. Shew that

$$1\pm 2n+3\,\frac{n\,(n-1)}{1.2}\pm 4\,\frac{n\,(n-1)\,(n-2)}{1.2.3}+\&c.=(n+2)\,2^{n-1}\text{ or }0,$$

according as we use the upper or lower signs.

202. Write down the general terms of

$$(1-x^2)^{-\frac{1}{2}},\quad (a^4-a^2x^2)^{\frac{7}{3}},\quad (a-x)^{-\frac{1}{3}}.$$

203. Given $\log\frac{1}{16}=\bar{2}.7958800$ and $\log 20.001=1.3010517$, find the logs of 2.000037 and .02000073: and determine the numbers whose logs are 3.3010395, .3010426, $\bar{2}.3010479$.

204. A farm is let for n years at a fixed rent and a fine of £P. When p years of the lease remain, what fine must be paid to extend these p years to q, at Compound Interest?

205. A and B are playing with two dice, each having staked 1s, the highest throw to win. A has thrown 6; what is B's expectation?

206. If $x = 1 + h$, where h is small, then (approximately)
$$\sqrt{\frac{2x - x^2 + a^2x^2}{a^2+1}} = 1 + \frac{a^2h}{a^2+1}, \text{ and } a^p x^{-m} + (b^q - a^p)\, x^{-n} = b^q x^{(n-m)\,a^p b^{-q} - n}.$$

207. Eliminate a and b from the equations
$$\frac{a^3 - x^3}{b^3 - y^3} = \frac{2x + 3y}{3x + 2y}, \quad a^3 - b^3 = (x - y)^3, \quad a^{\frac{3}{2}} + b^{\frac{3}{2}} = z^{\frac{3}{2}}.$$

208. S_1, S_2, S_3, &c. are the sums, to n terms, of n Geom. series, whose first terms are each unity, and common ratios, 1, 2, 3, &c.: shew that
$$S_1 + S_2 + 2S_3 + 3S_4 + \text{\&c.} + (n-1)\,S_n = 1^n + 2^n + 3^n + \text{\&c.} + n^n.$$

209. Shew that
$$\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \text{\&c. to } n \text{ terms} = \frac{n}{3\,(n+1)} + \frac{n}{6\,(n+2)} + \frac{n}{9\,(n+3)}.$$

210. There are three balls in a bag, one white, another black, and the third either white or black: if two be drawn, find the chance of their being (i) two black ones, (ii) one black and one white.

211. Shew that $\dfrac{1}{1 + x^{m-n} + x^{m-p}} + \dfrac{1}{1 + x^{n-m} + x^{n-p}} + \dfrac{1}{1 + x^{p-m} + x^{p-n}} = 1.$

212. If the difference of a and b be small, compared with either, then $\sqrt[m]{a} - \sqrt[m]{b} : \sqrt[n]{a} - \sqrt[n]{b} :: n\sqrt[n]{b^{n-1}} : m\sqrt[m]{a^{m-1}}$, nearly.

213. From $\log 1\frac{1}{5} = .0791812$ and $\log 2\frac{2}{5} = .3802112$, find the value of $\sqrt[5]{(3.6)^3} \times \sqrt[4]{\frac{1}{25}} \div \sqrt[3]{8\frac{49}{72}}$, given the *mant.* for 45323 and 45324 to be 6563186 and 6563282.

214. Resolve into partial fractions $\dfrac{a + bx + cx^2}{x^2\,(x^2 + a^2)}$.

215. Compare the chances of throwing a single ace in one trial with two dice and in two trials with three.

216. Find what value of b will make $b^2 - 4ac$ a complete square.

217. At a contested election, the n° of candidates was one more than the n° of persons to be elected, and each elector, by voting for one, two, &c. or as many as were to be elected, could dispose of his votes n ways. Find the n° of candidates.

218. In bringing an irreducible fraction to a circulating decimal, shew that, when any two rem^{rs} give the div^{r} for their sum, the two consecutive rem^{rs} will give the same sum, and the sum of the two figures in the period, which correspond to those rem^{rs}, will be 9.

219. Sum to n terms

(i) $\dfrac{1}{3.6} - \dfrac{1}{6.8} + \dfrac{1}{9.10} - \&c.$ (ii) $\dfrac{1}{1.2.4} + \dfrac{1}{2.3.5} + \dfrac{1}{3.4.6} + \&c.$

220. A, tossing a coin, is to pay B 1*s* if it fall *heads* the first time, 2*s* if the second, 3*s* if the third, and so on for n throws, the game to cease as soon as it falls heads. Find B's expectation.

221. Eliminate x, y, z, from the equations

$$\frac{a^m}{x^{m+n}} = \frac{b^m}{y^{m+n}} = \frac{c^m}{z^{m+n}}, \quad \frac{a^m}{x^m} + \frac{b^m}{y^m} + \frac{c^m}{z^m} = 1 = \frac{x^n + y^n + z^n}{k^n}.$$

222. Find the coefficients of x^3 and x^4 in $(1 + x^{\frac{1}{2}} + x^{\frac{3}{2}} + x^{\frac{5}{2}} - x^{\frac{7}{2}})^5$.

223. Find n integers in A.P. whose sum shall be n^2, whatever n may be.

224. If $\sqrt[n]{x}+m\sqrt[n]{y} : \sqrt[n]{x}-m\sqrt[n]{y} :: \sqrt[2n]{x}+m\sqrt[2n]{(x-y)} : \sqrt[2n]{x}-m\sqrt[2n]{(x-y)}$, shew that $x : y :: 1 \pm \sqrt{5} : 2$.

225. If on the average 9 ships out of 10 return safe to port, find the chance that out of 5 ships expected, at least 3 will arrive.

226. Find three square numbers whose sum shall be a given square (a^2). Ex. 81.

227. In Ex. 217, find the n° of candidates, if it was one more than twice the number of persons to be elected.

228. If A_m denote the middle term of $(1 + x)^{2m}$, then

$$A_0 + A_1 + A_2 + \&c. = (1 - 4x)^{-\frac{1}{2}}.$$

229. Sum $1 + \dfrac{1}{2}n + \dfrac{1}{3}\dfrac{n(n-1)}{1.2} + \&c.$, and (to n terms)

$$1.2^2 + 2.3^2 + 3.4^2 + \&c.$$

230. Shew that in taking a handful of shot from a bag, it is more probable that an odd number will be drawn than an even one.

231. If $ax^3 = by^3 = cz^3$, and $x^{-1} + y^{-1} + z^{-1} = k^{-1}$, then

$$(ax^2 + by^2 + cz^2)^{\frac{1}{3}} = (a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}})\, k^{\frac{2}{3}}.$$

232. Shew that, if $pq = r$, the equation $x^3 + px^2 + qx + r = 0$ will have two roots equal and of opposite signs.

233. Find three numbers with prime den^rs^, whose sum shall be $1\frac{86}{810}$.

234. Resolve into partial fractions

$$\frac{7x + 8}{(x^2 + 1)(x + 1)^2} \text{ and } \frac{7x + 8}{(x^2 + x + 1)(x + 1)^2}.$$

235. If the chance for an event A : that for B :: $m : n$, then in $r(m + n)$ trials it is most likely that A will happen rm times and B rn times.

236. If $aX + bY + cZ = 0$ and $a'X + b'Y + c'Z = 0$,
where $X = ax + a'x' + a''$, $Y = bx + b'x' + b''$, $Z = cx + c'x' + c''$,
then $X^2 + Y^2 + Z^2 = \dfrac{\{a''(bc' - b'c) + b''(a'c - ac') + c''(ab' - a'b)\}^2}{(bc' - b'c)^2 + (a'c - ac')^2 + (ab' - a'b)^2}$.

237. If from a vessel, containing a gallons of wine, b gallons be drawn off, and the vessel filled up with water, and this be repeated n times, find the quantity of wine remaining.

238. Shew that $\log_e x = \dfrac{m}{n}\{(1 - \sqrt[m]{x^{-n}}) + \frac{1}{2}(1 - \sqrt[m]{x^{-n}})^2 + \frac{1}{3}(1 - \sqrt[m]{x^{-n}})^3 + \&c.\}$.

239. Shew that $1 + 2^2 + 3 + 4^2 + 5 + 6^2 + \&c.$ to n terms
$$= \tfrac{1}{12}(n+1)(2n^2 + n + 3) \text{ or } \tfrac{1}{12}n(n+4)(2n+1),$$
according as n is odd or even.

240. A bets B 10s to 1s that he will throw heads at least once in three trials: what is B's expectation, and what would have been a fair bet?

241. Prove that
$$(Aa + Bb + Cc + \&c.)^2 = (A + B + C + \&c.)(Aa^2 + Bb^2 + Cc^2 + \&c.)$$
$$- AB(a-b)^2 - AC(a-c)^2 - BC(b-c)^2 - \&c.$$

242. If $x = 1$ nearly, then
$$mx^m - nx^n = (m-n)x^{m+n}, \text{ and } x^{-e} = \sqrt{\{1 - e^x + \sqrt[x]{e}\}} \text{ nearly}.$$

243. Find the coefficients of x^3, x^5, and x^7, in $(1 - 2x - 3x^2)^{-2}$.

244. Find an A. P., beginning with unity, in which the sum of the first half of any even number of terms shall have to the second half a constant ratio. Shew that there is but one such series.

245. Compare the chances of throwing two aces only in two trials with three dice and in three trials with four.

246. If $2x_r = y^{(2r-1)z} - y^{(1-2r)z}$, then $x_1(x_1 + x_2 + \&c. + x_{2n-1}) = x_n^2$.

247. Shew that $x^4 + px^3 + qx^2 + rx - s^2$ can be resolved into rational quadratic factors, if $s^2 = \dfrac{r^2}{p^2 - 4q}$; and hence solve the equation
$$x^4 - 6x^3 + 5x^2 + 8x - 4 = 0.$$

248. Given $x = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \&c.$ and $y = z\sqrt{(1 - y^2)}$, find y in terms of x by reversion of series.

249. Solve in positive integers $2xy - 3x^2 + y = 1$.

250. A is allowed to draw two coins from a bag, containing five sovereigns and four shillings. What is his expectation? and what if B draws a coin *before* or *after* A's first draw?

251. If $\frac{ad-bc}{a-b-c+d}=\frac{ac-bd}{a-b-d+c}$, then either of these fractions $=\frac{1}{4}(a+b+c+d)$.

252. If a, b, c, are in G.P., shew that $a^2-b^2+c^2>(a-b+c)^2$; and if $a^2+b^2+c^2=1=a'^2+b'^2+c'^2$, then $aa'+bb'+cc'<1$.

253. Find the fifth term of $(a^3+b^3\sqrt{-1})^{-\frac{2}{3}}$, the fifth and the greatest terms of $(1-\frac{3}{7})^{-\frac{8}{3}}$, and the fifth term of $(2-5x-7x^2)^5$.

254. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=$ &c., then $\frac{a^{3n}-c^{3n}}{b^{3n}-d^{3n}}=\frac{a^nc^ne^n-(a^n-c^n+e^n)^3}{b^nd^nf^n-(b^n-d^n+f^n)^3}$.

255. In Ex. 250, what will be A's chance in each case, if B's coin, being looked at, is found to be a sovereign, A *not* looking at his, till he has drawn them both?

256. Shew that $\left(\frac{x-x'}{a}\right)^2+\left(\frac{y-y'}{b}\right)^2+\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)=0$ resolves itself into the two equations $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$ and $\frac{x}{x'}=\frac{y}{y'}$.

257. Find three numbers such that the sum of all three and of every two may be squares.

258. If $$\frac{a+(a+y)x+(a+2y)x^2+\text{\&c. } ad\ inf.}{a+(a-y)x+(a-2y)x^2+\text{\&c. } ad\ inf.}=b,$$ and if x receive values in H.P., shew that the corresponding values of y will be in A.P.

259. Shew that the series $x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-$ &c. converges at the n^{th} term, if $n>\frac{1}{2}x$; and find the greatest term of $(x-x^{-1})^n$, when $x=2$, $n=5\frac{1}{2}$.

260. There are three black and four white balls in a bag, and three persons draw one each in succession, not replacing them. Find (i) each person's chance of drawing black, (ii) the chance of first and third drawing black and second white, (iii) of all three drawing black.

261. Find the value of $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$, if $x=\frac{12ab}{a+b+\sqrt{\{(a+b)^2+12ab\}}}$.

262. Resolve into factors of the first degree
$$2x^2-21xy-11y^2+34y-x-3.$$

263. Shew that $n^2-n+1:n^2+n+1$ lies between 3 and $\frac{1}{3}$.

264. If $\{f(x)\}^2=1+\{f'(x)\}^2$, and $a^x=f(x)+f'(x)$, shew that $f(x\pm y)=f(x)f(y)\pm f'(x)f'(y)$, and $f'(x\pm y)=f'(x)f(y)\pm f(x)f'(y)$.

265. From a bag containing 2 guineas, 3 sovereigns, and 7 shillings, A is allowed to draw three coins. What is his expectation? and what if one of the guineas should be known to be base?

266. Shew that $\frac{1}{85}\{3.10^x - 25(-1)^x\}$ is a positive integer, when x is so. Find the fifth term of the series of which it is the x^{th}, and sum the series to n terms.

267. Find a series of numbers which shall be at the same time of the forms $n^2 - 1$ and $10m^2$.

268. How small must x be taken so that the third term of $1 + 3x + 5x^2 + 7x^3 + \&c.$ may contain the sum of all that follow at least 500 times.

269. A person devised his estate among n persons in the following manner. A was to receive £P + 1-n^{th} of the remainder, B £$2P$ + 1-n^{th} of the remainder, C £$3P$ + 1-n^{th} of the remainder, and so on: find the value of the estate.

270. In one of two purses there are three sovereigns and a shilling, in the other three shillings and a sovereign. A coin is taken from one (it is not known which) and dropped into the other; and then, on drawing a coin from each bag, they are found to be two shillings. What is the chance that this will occur again, if two more are drawn, one from each purse?

271. Shew that $(n+1)(n+2)(n+3)\ldots$ to n factors $= 1.3.5\ldots(2n+1)2^n$; and if $a^1.a^3.a^5\ldots = p$, find the number of the factors a^1, a^3, &c.

272. If $\dfrac{a+bx}{a-bx} = \dfrac{b+cx}{b-cx} = \dfrac{c+dx}{c-dx} = \&c.$, and if also $a^x = b^y = c^z = \&c.$, then will a, b, c, &c. be in G.P., and x, y, z, &c. in H.P.

273. Find the coefficient of x^n in $(1 + x + 2x^2 + 3x^3 + \&c.)^2$.

274. What value of y will make $2(y^2+y)x^2 + (11y-2)x + 4$ and $2(y^3+y^2)x^3 + (11y^2-2y)x^2 + (y^2+5y)x + 5y - 1$ commensurable?

275. From a bag, containing $2n + 1$ balls, $2n$ are taken out, and are found to be alternately white and red. Shew that it is equally likely that the remaining one is either red or white; and find the chance that it is neither the one nor the other.

276. Shew that any triangle will have its area expressed in rational terms, if its sides be proportional to

$$gh(k^2 + l^2),\quad kl(g^2 + h^2),\quad (hk + gl)(hl - gk).$$

277. If p be prime, and neither a nor $a-1$ a mult. of p, and m a positive integer, then each of the sums $a^{m+1} + a^{m+2} + \&c. + a^{m+p-1}$ and $a^{m+1} + a^{m+2} + \&c. + a^{mp}$ is a multiple of p.

278. Sum to n terms and *ad inf.* $\dfrac{2}{3.5} - \dfrac{3}{5.7} + \dfrac{4}{7.9} - \dfrac{5}{9.11} + \&c.$

279. If one of the roots of $x^2 - px + q = 0$ be large, compared with the other, shew that p, $p - \frac{q}{p}$, $\frac{p(p^2 - 2q)}{p - q}$, are closer and closer approximations to it.

280. A is to receive a certain number of farthings, expressed, he knows, by the digits 1, 2, 3, 4, 5, but in what order he is not aware. Find his expectation.

281. What value of x makes $(4x + 1)(2z + 1)^2 = 5(3x + 1)(z + 3)^2$ for large values of z? What values of x and y make the fraction $\frac{2z^2 + (x - a)z + 2b(x - 2c)}{3z^2 + (y - b)z + 3a(y - 3c)}$ independent of z?

282. If $x + c$ be the G. C. M. of $x^2 + ax + b$ and $x^2 + a'x + b'$, their L. C. M. will be $x^3 + (a + a' - c)x^2 + (aa' - c^2)x + (a - c)(a' - c)c$.

283. Extract the square roots of $x^m + \frac{1}{4}\sqrt[np]{(b^{4p}x^{4n})} - \sqrt[n]{b}.\sqrt[2p]{x^{mp+4}}$, and of $4\{(a^2 - b^2)cd + ab(c^2 - d^2)\}^2 + \{(a^2 - b^2)(c^2 - d^2) - 4abcd\}^2$.

284. Find the maximum or minimum value of $\frac{a^2}{x^2} + \frac{b^2}{a^2 - x^2}$.

285. There are three white and five black balls in a bag, and three persons draw a ball in succession (the balls drawn not being replaced) until a white one is drawn: shew that their respective chances are as 27 : 18 : 11.

286. Find two numbers such that, if their sum be added to the square of each, the results shall be squares.

287. Sum to n terms and *ad inf.*, when $x < 1$, the series

$$1 + 2x^2 + 2x^3 + 6x^4 + 10x^5 + 22x^6 + 42x^7 + 86x^8 + \&c.;$$

and find also the coefficient of x^n.

288. Shew that the series for $(1 - x)^{-n}$ diverges or converges from the first term, according as $x \gtrless 1$. From what term does the series for $(1 + x)^n$ converge?

289. If n be a prime number and N a number prime to n, then, when the square numbers N^2, $4N^2$, $9N^2$, &c., $\{\frac{1}{2}(n - 1)\}^2 N^2$ are divided by n, they will each leave a different remainder.

290. Find the chance that, if a halfpenny be tossed, it will neither fall heads nor tails three times successively in five trials, but will fall heads the sixth and tails the three following times.

291. Shew that $\frac{cd - ab \pm \sqrt{\{(a - c)(a - d)(b - c)(b - d)\}}}{a + b - c - d}$ is always a possible quantity, if a, b, c, d, are the roots of a biquadratic with rational coefficients.

292. Given $2(x^2 + y^2 - x - y) + 1 = 0$, find x and y: and given $\sqrt{x} + \sqrt{y} : \sqrt{(2x)} - \sqrt{(3y)} :: a : b$, find the value of

$$\sqrt{x} - \sqrt{y} : \sqrt{(2x)} + \sqrt{(3y)}.$$

293. If the difference between the $(n-1)^{\text{th}}$ and n^{th} terms of an H. P. be $\dfrac{1}{an^2 + bn + c}$, find the relation between a, b, and c.

294. Solve in positive integers $2xy - 3x^2 + y = 1$.

295. An even n° (n) of pieces of money being thrown, shew that it is $2^{n-1}+1$ to $2^{n-1}-1$ against there being an even n° of heads.

296. A person spends in the first year m times the interest of his property, in the second $2m$ times that of the remainder, in the third $3m$ times that of what is now remaining, and so on; and at the end of $2p$ years has nothing left. Shew that in the p^{th} year he spends as much as he had left at the end of that year.

297. If N be any number, which differs from the square numbers next greater and less than it by a and b respectively, prove that $N - ab$ is a square number.

298. Find two integers such that if unity be added to each of them, as also to their sum and difference, the four results shall be squares.

299. If $x = 1 + n^{-1}$, shew that the sum of n terms of the series $1 + 2x + 3x^2 +$ &c. is n^2.

300. Shew that it is *probable* that, in 25 throws with two dice, sixes will be thrown at least once.

301. Shew that 12345654321 is divisible by 12321 in any scale, where the radix exceeds 6.

302. Find the greatest term in the expansion of $(\sqrt{2} + \sqrt{3})^{12\sqrt{6}}$.

303. Find two integers, such that the sum or difference of their squares shall each exceed unity by a square number.

304 Sum to n terms and *ad infinitum*, when $x < 1$,

$$1 + 8x + 27x^2 + 64x^3 + 125x^4 + 216x^5 + 343x^6 + \&\text{c}.$$

305. Four cards being drawn from a common pack, find the chance that they are marked one, two, three, four, of different suits.

306. If $\phi(n, m) = \dfrac{1}{m} - n\dfrac{1}{m+p} + \dfrac{n(n-1)}{1.2}\dfrac{1}{m+2p} -$ &c., shew that $\phi(n, m) = \dfrac{np}{m}\phi(n-1, m+p)$; and thence deduce the sum of the series, when n is a positive integer.

307. Shew that if any number, N, can be resolved into the sum of n squares, then $2(n-1)N$ can be resolved into the sum of $n(n-1)$ squares.

308. If $z^2+z+1=0$, shew that the sum of those terms of the expansion of $(1+x)^n$, in which the index of x is a mult. of 3,
$$=\tfrac{1}{3}\{(1+xz)^n+(1+x)^n+(1+xz^{-1})^n\}.$$

309. Sum to n terms and *ad infinitum* $\dfrac{2}{1.3.3}+\dfrac{3}{3.5.9}+\dfrac{4}{5.7.27}+\&c.$

310. If p witnesses concur in their statement of a fact, which they have heard from another individual, shew that the chance of its being true is $\dfrac{v^p-mv^{p-1}+m^2v^{p-2}-\&c.}{v^p+m^p}$, where v = veracity of each of the $p+1$ persons, and $m=1-v$.

311. Shew that $n>\log_e(1+n)$, and that $4xy-3(x^2-y^2)^{\frac{2}{3}} \gtrless 1$, according as $(x+y)^{\frac{2}{3}}-(x-y)^{\frac{2}{3}} \gtrless 1$.

312. If p_r be the coeff. of x^r in $(1+x)^n$, (n a positive int.,) shew that
$$\frac{p_1}{p_0}+2\frac{p_2}{p_1}+3\frac{p_3}{p_2}+\&c.+n\frac{p_n}{p_{n-1}}=\frac{n(n+1)}{2},$$
and $(p_0+p_1)(p_1+p_2)(p_2+p_3)\dots(p_{n-1}+p_n)=\dfrac{(n+1)^n}{\lfloor n}\,p_1p_2p_3\dots p_n.$

313. A circular field is divided into an odd number of equal areas by the circumferences of concentric circles; and the radius of the outer circle is p times the breadth of the middle area. Find the number of circles.

314. Eliminate x, y, z from the equations
$$x^2(y+z)=a^3,\ y^2(x+z)=b^3,\ z^2(x+y)=c^3,\ xyz=abc.$$

315. A's skill is to B's as $1:3$, to C's as $3:2$, and to D's as $4:3$; find the probability that A in three trials, one with each person, will succeed (i) twice *exactly*, (ii) twice *at least*.

316. If $f(p, q)$ denote the coeff. of x^q in $(1+x+x^2+\&c.+x^n)^p$, shew that $f(1,q)-\tfrac{1}{2}f(2,q-1)+\tfrac{1}{3}f(3,q-2)-\&c.=\dfrac{1}{q+1}$, unless $q+1$ should be a multiple of $n+2$.

317. If a_1, a_2, &c. are in G. P., and x_1, x_2, &c. in A. P., shew that
$$a_1^{x_1}\cdot a_2^{x_2}\dots a_n^{x_n}=\{a_1^{(2n-1)x_1+(n-2)x_n}\times a_n^{(2n-1)x_n+(n-2)x_1}\}^{\frac{n}{6(n-1)}}.$$

318. Sum to n terms and *ad infinitum* each of the series,
$$\frac{5}{1.2.3.2}+\frac{6}{2.3.4.4}+\frac{7}{3.4.5.8}+\&c. \text{ and } \frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\&c.$$

319. Investigate the general forms of x and y, which rationalize $(ax^2 + by^2)^{\frac{1}{n}}$, when $n = 3$ or any *odd* number.

320. Fifteen persons sit down at a round table. Shew that it is 6 to 1 against two particular persons sitting next each other; and that, generally, for n persons, the odds against the same event are $n - 3 : 2$.

321. If $\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}, \frac{p_{n+1}}{q_{n+1}}$, be consecutive convergents to x, then each of the fractions $\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \frac{2p_n + p_{n-1}}{2q_n + q_{n-1}}, \frac{3p_n + p_{n-1}}{3q_n + q_{n-1}}$, &c. $\frac{p_{n+1}}{q_{n+1}}$, approach more and more nearly to the value of x than any fraction with smaller denominator.

322. If a, b, c, &c. k, be n unequal numbers, and $m < n - 1$, then

$$\frac{a^m}{(a-b)(a-c)...(a-k)} + \frac{b^m}{(b-a)(b-c)...(b-k)} + \&c. + \frac{k^m}{(k-a)(k-b)...} = 0.$$

323. Shew that $\log_e x = (x - 1) \frac{2}{x^{\frac{1}{2}} + 1} \cdot \frac{2}{x^{\frac{1}{4}} + 1} \cdot \frac{2}{x^{\frac{1}{8}} + 1} \ldots$ *ad inf.*

324. Find two integers such that their difference, the diff. of their squares, and the diff. of their cubes, may all be squares.

325. The corners of a common die are filed away, till the faces, which before were squares, become regular octagons. Compare the chances of its falling, when thrown, upon a triangular or octagonal face, neglecting all mechanical considerations.

326. If $s = a^2 + b^2$, $p = 2ab$, $P = (a+b)^p$, shew that $P.P^{\frac{1}{2}}.P^{\frac{1}{4}}.P^{\frac{1}{8}} \ldots$ *ad inf.*

$$= s^p + p^2 s^{p-1} + p^3 . \frac{p-1}{2} s^{p-2} + p^4 . \frac{p-1}{2} . \frac{p-2}{3} s^{p-3} + \&c.$$

327. Sum *ad infinitum*

$$\frac{1^2}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \&c., \quad \frac{1^2}{3} + \frac{2^2}{3^2} + \frac{3^2}{3^3} + \&c., \quad \frac{1^2}{5} - \frac{2^2}{5^2} + \frac{3^2}{5^3} - \&c.$$

328. How many triangular pyramids can be formed, whose edges are six given lines, any two of which are > than the third?

329. Eliminate m, n, p, q from the equations

$$\frac{x - p}{m} + \frac{y + q}{n} = \frac{pm}{a^2} + \frac{qn}{b^2} = \frac{m^2}{a^2} - \frac{n^2}{b^2} = \frac{p^2}{a^2} + \frac{q^2}{b^2} - 1 = 0.$$

330. Find the *probable* sum of the series $1 + x + x^2 +$ &c., when the number of its terms is known to be not greater than q nor less than p.

331. If $x = 1$ nearly, shew that x, $1 - x + x^2$, $\frac{1}{2}(1 + x - x^2 + x^3)$, are nearer and nearer approximations to the value of x^x.

332. If y be the Harmonic mean between x and z, and x and z respectively the Arithmetic and Geometric means between a and b, shew that $y = 2(a+b) \div \left\{\left(\frac{a}{b}\right)^{\frac{1}{4}} + \left(\frac{b}{a}\right)^{\frac{1}{4}}\right\}$.

333. Of $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{\&c.}$, $1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \text{\&c.}$, the former series is *divergent* and the latter *convergent*.

334. If $x \div \{(1-x)^2 - cx\}$ be expanded in a series of ascending powers of x, shew that the coefficient of x^r is

$$r\left\{1 + \frac{r^2-1}{2.3}c + \frac{(r^2-1)(r^2-4)}{2.3.4.5}c^2 + \frac{(r^2-1)(r^2-4)(r^2-9)}{2.3.4.5.6.7}c^3 + \text{\&c.}\right\}.$$

335. There are a n° of tickets, marked with some one of the numbers from 1 to n^2+1. Every one of these numbers, (such as r,) is marked upon the same number (r) of tickets; and every ticket marked with a square number (m^2) confers a prize of m shillings. *A* draws one ticket: find his expectation.

336. Shew that the product of any number of factors of the form $x^2 + axy + by^2$, $x'^2 + ax'y' + by'^2$, &c., may be put into the same form, $X^2 + aXY + bY^2$.

337. Apply the preceding to find a series of positive integral values for x and y, which shall make $x^2 + 3xy + 5y^2$ a square number.

338. If $s = a + b$, $p = ab$, and $q = a \div b$, prove that

$$p^2 = s^4\left(q^2 - 4q^3 + \frac{4.5}{1.2}q^4 - \frac{4.5.6}{1.2.3}q^5 + \text{\&c.}\right),$$

and $a^n + b^n = s^n - nps^{n-2} + \frac{1}{2}n(n-3)p^2s^{n-4} - \text{\&c.}$

339. Sum to n terms and *ad infinitum* $\frac{1}{3.8} + \frac{1}{6.12} + \frac{1}{9.16} + \text{\&c.}$

340. A bag contains 50 balls, 5 of which are drawn at a time, and replaced after each drawing. Two persons draw alternately, the prize being won by him who first draws two particular balls. Find the odds in favour of the first drawer?

341. $$\frac{1}{\log_e x} = \frac{1}{2} \cdot \frac{x+1}{x-1} - \frac{1}{2}\left\{\frac{1}{2} \cdot \frac{\sqrt{x}-1}{\sqrt{x}+1} + \frac{1}{4} \cdot \frac{\sqrt[4]{x}-1}{\sqrt[4]{x}+1} + \frac{1}{8} \cdot \frac{\sqrt[8]{x}-1}{\sqrt[8]{x}+1} + \text{\&c.}\right\}.$$

342. Shew that every term of the preceding series (within the brackets) is greater than $\frac{1}{4}$th of the term next before it, and less than a third proportional to the two next before it.

343. Shew that the sum of the products of n quantities c, c^2, c^3, &c. taken m and m together, is $c^{\frac{m(m+1)}{2}} \times \frac{(c^n-1)(c^{n-1}-1)\dots(c^{n-m+1}-1)}{(c-1)(c^2-1)\dots(c^m-1)}$.

344. If a, b, c, &c. are any n quantities, shew that $a^n + b^n + c^n + \&c. > n(abc\ldots)$; and thence prove that $1.2.3\ldots n < \{\frac{1}{2}(n+1)\}^n$.

345. A plays at a game, in which he reckons his chance of success to be e. If it be an even chance that he has made an error e', or not, in his calculation, shew that this does not affect his chance of success in a single trial, but increases his chance of continual success in any number of repeated trials.

346. Eliminate x from $x^4 + rx + s = 0$, $x^2 + vx = y$; and shew that the equation in y will be a biquadratic, wanting its second and fourth terms, if $rv^3 + 4sv^2 - r^2 = 0$. Ex. Solve $x^4 + 3x = 2$.

347. If n be a prime n°, the expression $A_0x^m + A_1x^{m-1} + \&c. + A_m$, cannot admit of more than m different values of x, less than n, which will render it divisible by n.

348. Apply the preceding to shew that, if n be a prime number, each of the quantities $N^{\frac{n-1}{2^p}} \pm 1$ has $\dfrac{n-1}{2^p}$ integral values of N less than n, which make it divisible by n.

349. Shew that the number of different ways, in which the letters of the expression $p^{m+r}q^n$ can be written at length, so that at least r p's may always follow each other, is $(n+1)\dfrac{\lfloor m+n}{\lfloor m \lfloor n}$.

350. Apply the preceding to find the chance that, on tossing a shilling 12 times, it will fall heads at least 6 times successively.

351. Prove that, when the expression in Ex. 347 admits of exactly m such different values of x, that render it a multiple of n, then the quantities $A_0S_1 + A_1$, $A_0S_2 - A_2$, &c. $A_0S_m - (-1)^m A_m$, are all multiples of n, where S_r denotes the sum of the products of those values of x, taken r together.

352. Apply the preceding to shew that, when n is a prime number, $1.2.3\ldots(n-1)+1$ and $1.2.3\ldots(n-1)\left(1 + \frac{1}{2} + \frac{1}{3} + \&c. + \dfrac{1}{n-1}\right)$ are each multiples of n.

353. The chance of an event, whose prob^y is p, occurring at least r times successively in $n + r$ trials, is $\{1 + n(1-p)\}p^r$.

354. Two persons are known to have passed over a piece of road in opposite directions within the time $a + b + c$, in the times a and b respectively. Find the chance that they will meet.

EXAMPLES: Part III.

EQUATION PAPERS OF ST. JOHN'S COLLEGE, CAMBRIDGE.

1.

1. $\dfrac{3x+y}{14} - \dfrac{2x-7}{21} + 2\tfrac{3}{4} = \dfrac{x-4}{4}$

2. $\left.\begin{aligned} 3x + 6y + 1 &= \dfrac{6x^2 + 130 - 24y^2}{2x - 4y + 3} \\ 3x - \dfrac{151 - 16x}{4y - 1} &= \dfrac{9xy - 110}{3y - 4} \end{aligned}\right\}$

3. $\dfrac{x+2}{x-1} - \dfrac{4-x}{2x} = 2\tfrac{1}{8}$

4. $5y + \tfrac{1}{5}\sqrt{(x^2 - 15y - 14)} = \tfrac{1}{3}x^2 - 36$
$\dfrac{x^2}{8y} + \dfrac{2x}{3} = \sqrt{(\dfrac{x^3}{3y} + \dfrac{x^2}{4})} - \dfrac{y}{2}$

5. A brewer, from ingredients worth £20, brews 500 gallons of ale, (on which there is a duty of 6*d* *per* gallon,) and sells it at 2*s* a gallon. From the same ingredients he afterwards brews the same n° of gals, part strong beer, (on which he pays ale-duty,) and the rest small beer, (on which he pays $\frac{1}{4}$ ale-duty.) By mixing, and selling the mixture as ale, his gains are increased in the ratio of 10 : 7. Find the n° of gals of strong beer.

6. The n° of deaths in a besieged garrison was 6 daily, and, allowing for this, their provisions would just have lasted 8 days. But on the 6th evening, 100 men were killed in a sally, and afterwards the mortality increased to 10 daily. At the end of the 6th day there was stock enough remaining to support 6 men for 61 days: how many will be alive when the whole is exhausted?

7. A man buys a guinea at the market-price of standard gold; but an Act passing, which makes it illegal to sell the coin of the realm, he clips off $\frac{1}{25}$th part. He may now legally sell it as a light guinea; and finds that by reason of the rise of pure gold in the ratio of 239 : 249, he just gains the clippings by his purchase. Find the ratio of pure gold and alloy in the guinea, and also the relative value of equal quantities of pure gold and alloy, having given that the sum of the squares of the two ratios exceeds eleven times their sum by $233\frac{9}{121}$.

2

1. $$\frac{2x+1}{29} - \frac{402-3x}{12} = 9 - \frac{471-6x}{2}$$

2. $$\left.\begin{aligned} &\tfrac{1}{3}(3x-5y) - \tfrac{1}{12}(2x-8y-9) = \tfrac{1}{2}y + \tfrac{1}{3} + \tfrac{1}{4} \\ &\tfrac{1}{7}x + \tfrac{1}{4}y + 1\tfrac{1}{3} : 4x - \tfrac{1}{8}y - 24 :: 3\tfrac{1}{3} : 3\tfrac{1}{2} \end{aligned}\right\}$$

3. $$\frac{x+3}{2} + \frac{16-2x}{2x-5} = 5\tfrac{1}{6}$$

4. $$\left.\begin{aligned} &\frac{x+y+\sqrt{(x^2-y^2)}}{x+y-\sqrt{(x^2-y^2)}} = \frac{9(x+y)}{8y} \\ &(x^2+y)^2 + x - y = 2x(x^2+y) + 506 \end{aligned}\right\}$$

5. At the review of an army, the troops were drawn up in a solid mass, 40 deep, when there were just $\frac{1}{4}$ as many men in front as there were spectators. Had the depth, however, been increased by five, and the spectators drawn up with the army, the n° of men in front would have been 100 fewer than before. Find the force of the army.

6. A n° of persons bought a field for £345, the youngest paying a certain sum, the next £5 more, and so on in A. P. The younger half took a portion of the field proportional to the sum they had subscribed; and this they agreed to divide equally, by equalizing their contributions to £22 each. How many persons were there in all?

7. *A* and *B* travelled on the same road and at the same rate from H to L. At the 50th milestone from L, *A* overtook a drove of geese, which were proceeding at the rate of 3 miles in 2 hrs; and 2 hrs afterwards he met a waggon, moving at the rate of 9 miles in 4 hrs. *B* overtook the geese at the 45th milestone, and met the waggon just 40 minutes before he came to the 31st milestone. How far was *B* from L when *A* reached it?

3.

1. $$\frac{4x-34}{17} - \frac{258-5x}{3} = \frac{69-x}{2}$$

2. $$\left.\begin{aligned} &\tfrac{1}{3}(4x-2y+3) - \tfrac{1}{7}(18-x+5y) = \tfrac{1}{4}x - \tfrac{1}{6}y - \tfrac{1}{7} - 7\tfrac{7}{10} \\ &2x - y + 15 : y - 2x + 15 :: \tfrac{1}{3}x - \tfrac{1}{4}y + \tfrac{3}{4} : \tfrac{1}{4}y - \tfrac{1}{3}x + \tfrac{1}{12} \end{aligned}\right\}$$

3. $\dfrac{2x}{x-4}+\dfrac{2x-5}{x-3}=8\frac{1}{3}$

4. $\left.\begin{array}{l}\dfrac{y}{x}-\dfrac{9\sqrt{x}}{y}-\dfrac{81}{xy}=(2y+9)\dfrac{\sqrt{x}}{y} \\ \dfrac{\sqrt{y}}{x}+3\sqrt{\dfrac{x}{y}}=\dfrac{9}{x\sqrt{y}}+\sqrt{x}\end{array}\right\}$

5. A packet from Dover reaches Calais in 2 hrs; but, on the return voyage, proceeds at first 6 miles an hour slower than it went. The wind, however, changing halfway, it sails 2 miles an hour faster, and reaches Dover sooner than it would have done, if the wind had not changed, in the proportion of 6 : 7. Find the distance between Dover and Calais.

6. From the middle of a town two streets branched off, and crossed a straight river by bridges *A* and *B*. From their junction, a sewer, equally inclined to both streets, led to a point in the river distant 6 chains from *A*, and from *B* 11 chains less than the length of the sewer, the expense of making which was as many £'s per chain as there were chains in the street leading to *A*. The sewer proving insufficient, a drain was made from a point in this street, distant 4 chains from *A*, which entered the river at the same point with the sewer, and was equally inclined to the river and sewer. Now a drain down each street, at £9 per chain, would have cost only £54 more than the sewer. Find the lengths of the streets and sewer.

7. A labourer, with his wife and children, saved each a certain n° of pence in A.P. The whole monthly saving was less by 3*s* 3*d* than the cost of $\frac{1}{6}$ as many bushels of wheat, as the seventh child saved pence, the price of wheat being such that the savings of the eldest and fifth child, increased by 10*s*, would buy two bushels. But wheat rising 2*s* a bushel, and work being scarce, they find that their savings will not buy as much wheat as before by two bushels; and that, at this rate, their annual savings would be less by 5 guineas than before. At this time, the two youngest died; and it was calculated that, if the others saved each 1*s* less than the eldest child had done before the rise of wheat, their monthly savings would not be affected by this event. How many were they in family?

4.

1. $\frac{1}{2}(5x-1) - \frac{1}{10}(7x-2) = 6\frac{3}{5} - \frac{1}{2}x$

2. $\left.\begin{aligned} &\sqrt{y} - \sqrt{(a-x)} = \sqrt{(y-x)} \\ &\sqrt{(y-x)} + \sqrt{(a-x)} : \sqrt{(a-x)} :: 5 : 2 \end{aligned}\right\}$

3. $\dfrac{5(3x-1)}{1+5\sqrt{x}} + \dfrac{2}{\sqrt{x}} = 3\sqrt{x}$

4. $\left.\begin{aligned} &\frac{y}{2x} + \frac{2}{3} \cdot \frac{y - \sqrt{(x-1)}}{y^2 - 2\sqrt{(x^2-1)}} = \frac{\sqrt{(x+1)}}{x} \\ &\tfrac{1}{2}y^4 = y^2x - 1 \end{aligned}\right\}$

5. A farmer laid up a stock of corn, expecting to sell it in 6 months at 3*s* a bushel more than he gave. But by that time, instead of having risen, corn had fallen 1*s* a bushel, and he found that he should, by selling now, lose the cost price of 5 bushels. He therefore kept it to the year's end; and then, being obliged to sell at 2*s* a bushel under prime cost, lost just 10*s* less than he had expected to gain. Find the cost price per bushel, allowing 5 per cent simple interest.

6. A ship, with a crew of 175, set sail with water enough to last the voyage. But, at the end of 30 days, the scurvy began to carry off 3 men daily, and a storm protracted the voyage 3 weeks. They just reached port, however, without the water falling short. Required the time of passage.

7. The hold of a vessel partly full of water, (which is uniformly increased by a leak,) has two pumps worked by *A* and *B*, of whom *A* takes 3 strokes to 2 of *B*'s, but 4 of *B*'s throw out as much water as 5 of *A*'s. *B* works for the time in which *A* alone would have emptied the hold; *A* then pumps out the rest, and the hold is cleared in $13\frac{1}{3}$ hours. Had they worked together, the hold would have been emptied in $3\frac{3}{4}$ hours, and *A* would have pumped 100 gals more than he did. Find the influx per hour at the leak.

5.

1. $\frac{1}{7}(4x-21) + 7\frac{5}{6} + \frac{1}{3}(7x-28) = x + 3\frac{3}{4} - \frac{1}{8}(9-7x) + \frac{1}{12}$

2. $\left.\begin{aligned} &\tfrac{1}{3}(3x-2y) + 1 + \tfrac{1}{8}(11y-10) = \tfrac{1}{7}(4x-3y+5) + \tfrac{1}{5}(45-x) \\ &45 - \tfrac{1}{3}(4x-2) = \tfrac{1}{18}(55x+71y+1) \end{aligned}\right\}$

3. $x^{\frac{7}{3}} + \dfrac{41\sqrt[3]{x}}{x} = \dfrac{97}{\sqrt{x^2}} + x^{\frac{5}{6}}$

4. $\left.\begin{array}{l} \dfrac{x^2y^2}{2} + 4 - 40y^2 = 140 - y^2\sqrt{(x^2 - \dfrac{272}{y^2})} \\ x^2 - \dfrac{2}{y}(\dfrac{3}{y} + 15x) = \dfrac{30}{y^2} + \dfrac{5x}{y} \end{array}\right\}$

5. On Jan. 1, 1799, a beggar received from *A* as many groats as *A* was years old, and a similar donation in each of the seven following years, during the last of which *A* died, his alms having in all amounted to £7 18*s* 8*d*. In what year was he born?

6. *A* entered into a canal speculation with 14 others, the profits of which were in all £595 more than five times the price of each share. Seven of his partners afterwards joined him in a scheme for navigating the canal with steam-boats, each venturing a sum less than his former gains by £173. But this concern failed, and *A* lost £419 by it; for they not only never recovered their outlay, but lost all their former gains and £368 besides. Find the cost price of shares in each concern.

7. *A*, *B*, *C* were architects. *A* and *B* built four warehouses with flat roofs, each a large and each a small one, the width of the large ones being the same, and likewise of the small ones. *A* built his as long and as high as they were broad; but *B* made the length and height of his small one the same as the breadth of his large one, and the difference between the contents of *A*'s and *B*'s buildings was 73728 feet. *C* also built upon a square plot of ground, whose area was the difference of those on which *A* built; and would have been in fact just 2688 square feet, if he had added to it eight times as many square feet as there were feet in its width. Find the width of the buildings.

6.

1. $\dfrac{2x}{3} - \dfrac{1 - \frac{1}{2}x}{4x} = \dfrac{x-1}{2} + \dfrac{x}{6} + \dfrac{7}{12}$

2. $\left.\begin{array}{l} x + 2y + 3z = 17 \\ 2x + 3y + z = 12 \\ 3x + y + 2z = 13 \end{array}\right\}$

3. $a^2b^2x^{\frac{1}{n}} - 4(ab)^{\frac{3}{2}}x^{\frac{m+n}{2mn}} = (a-b)^2x^{\frac{1}{m}}$

4. $\left.\begin{array}{l} x^4 + y^4 = 1 + 2xy + 3x^2y^2 \\ x^3 + y^3 = 2y^2x + 2y^2 + x + 1 \end{array}\right\}$

5. A can walk forwards four times as fast as he can backwards, and undertakes to walk a certain distance ($\frac{1}{4}$ of it backwards) in a certain time. But the ground being bad, he finds that his rate per hour backwards is $\frac{1}{6}$ of a mile less than he had reckoned, and that to win his wager he must walk forwards two miles an hour faster. What is his rate per hour backwards?

6. A lent B a sum at a certain rate of Int., taking as security such an amount of Spanish 5 *per cents* as would produce the same interest as the debt. At the year's end, B proved insolvent; and Spanish bonds having fallen 40 per cent, A lost £400. Had they not fallen, he would have been repaid with a surplus of £250; and if he had been at liberty to have sold them out at the half-year's end, when they were at 50, (which was before the int. upon them was due,) he would have lost only £300. Required the amount of the debt.

7. The builder of a treadmill agreed to take for payment the produce of n weeks' labour of the convicts then in prison, and to supply them with food during that period. His estimate was a shillings for the weekly expense of each man: but, finding this sufficient for the first week only of a man's labour, he increased it the second week by ra, the third, in addition, by r^2a, and so on. Now at the beginning of each week after the first, c fresh convicts were sent to the mill, when he found that, had his contract included these, he would have gained as much as he had calculated on at first. Supposing each man's weekly labour to be worth pa shillings, find the number of convicts at first.

7

1. $$\frac{11x-13}{25}+\frac{19x+3}{7}-\frac{5x-25\frac{1}{3}}{4}=28\tfrac{1}{7}-\frac{17x+4}{21}$$

2. $$\frac{3}{x}-\frac{4}{5y}+\frac{1}{z}=7\tfrac{3}{5},\quad \frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}=10\tfrac{1}{6},\quad \frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}=16\tfrac{1}{10}$$

3. $$\frac{9\frac{3}{5}-\frac{3}{2}\sqrt{x}-x}{5\sqrt{x}-8}+\frac{31}{50}=\frac{4}{5}\cdot\frac{\frac{74}{5}\sqrt{x}-x^{\frac{3}{2}}}{4x-7}$$

4. $$\left.\begin{aligned}&\left(\sqrt{\frac{x}{y}}-\sqrt{\frac{y}{x}}\right)^2+\sqrt{y}\left(\sqrt{x}-\frac{\sqrt{y}}{2}\right)=\frac{x}{\sqrt{y}}\left(\sqrt{x}-\frac{y}{\sqrt{2}}\right)\\ &9\sqrt{\frac{x}{y}}+3\sqrt{\frac{y}{x}}=\frac{21\sqrt{(2x)}-1}{2}\sqrt{\frac{y}{x}}+\frac{1}{2\sqrt{(xy)}}\end{aligned}\right\}$$

5. *A* sends his agent money to buy pimento, calculating the price at £8 for 5 bags. But the price having risen, the money sent would not buy as much by 18 bags as *A* intended: and it was found that $5\frac{1}{4}$ bags more than $\frac{1}{3}$ of the original quantity would now cost just £10 7*s* more than before. How many bags were purchased?

6. *A* and *B* set out from C and D, *A* starting 3 hrs before *B*. They meet at 20 miles from D, and *A* reaches D one hour before *B* reaches C. The next day *B* starts early, meeting *A*, who had then gone $\frac{1}{7}$ of his journey back; and, though delayed 3 hours, *B* reaches D in time to have gone 28 miles further before *A* reaches C. Required their rates per hour of journeying.

7. *A* called a meeting of his creditors, whose claims increased in A.P.; when it was found that his effects would have paid as many shillings in the £, as there were £'s in the common difference of the claims, and would in fact have exactly sufficed to pay the claims of the *third* and the *highest* creditors. But the latter failed to make good his claim, and the others consequently received 2*s* 8*d* in the £ more than they would have done; and the third and highest dividends were in consequence increased together by £9 12*s*. Find the assets.

8.

1. $$\frac{2x+8\frac{1}{3}}{9} - \frac{13x-2}{17x-32} + \frac{x}{3} = \frac{7x}{12} - \frac{x+16}{36}$$

2. $$\left.\begin{aligned} \frac{9}{8}\,\frac{\sqrt[3]{(x+y)}}{y} + \frac{9}{8}\,\frac{\sqrt[3]{(x+y)}}{x} &= 1\tfrac{1}{7} \\ \frac{7}{4}\,\frac{\sqrt[3]{(x-y)}}{y} - \frac{7}{4}\,\frac{\sqrt[3]{(x-y)}}{x} &= \frac{1}{9} \end{aligned}\right\}$$

3. $$\sqrt[2pq]{x^{p+q}} - \frac{1}{2}\,\frac{a^2-b^2}{a^2+b^2}\{\sqrt[p]{x} + \sqrt[q]{x}\} = 0$$

4. $$\left.\begin{aligned} 3x - x\sqrt{(\tfrac{5}{4}x^2 - 2y + 8)} &= 2 - y \\ \frac{\sqrt{(x+y)}}{2x} - \frac{3}{4}x &= \frac{2x-3}{\sqrt{(x+y)}} - \frac{3y}{2x} \end{aligned}\right\}$$

5. A farmer's rent was £50, and his expenditure (of which $\frac{1}{6}$ was in payment of assessed taxes) was such that he could only pay his landlord £30. The next year his rent was lowered 20 per cent, the taxes also were reduced one-half, and farm produce

increased in value one-third; and now, after paying his rent and former debts, he had £5 over. Find his expenditure.

6. *A* starts from Newmarket to London at the same time that *B* and *C* leave Hockeril and London for Newmarket. *A* meets *B* 4 hours before *C* overtakes *B*; but, on his return, *A* meets *C* one hour before he meets *B*, on their return also, all three having rested the same time at their destinations. Now *A* rode 10 miles an hour, and met *B* at the same place going and returning. Find the distance from London to Newmarket, Hockeril lying midway between them.

7. Two vessels, *P* and *Q*, contain fluids in the ratio of 4 : 21, which consist of different mixtures of wine and spirits. *A* pumps out of *P* into *Q*, and then *B* pumps into *Q* $\frac{3}{4}$ of what remains, and now the mixture *Q* is found to have only $\frac{12}{13}$ of its original strength. Now if, when *A* stopped, *B* had pumped as much as before from *Q* into *P* instead of from *P* into *Q*, the strength of *P* would have been a mean proportional between the original strengths of *P* and *Q*; and *B* would have pumped the same quantity of wine as before of spirits. Compare the quantities of fluid pumped by *A* and *B*, the strength of spirits being three times that of wine.

9.

1. $$\frac{25-\frac{1}{3}x}{x+1}+\frac{16x+4\frac{1}{3}}{3x+2}=5+\frac{23}{x+1}$$

2. $$\left.\begin{aligned}\frac{4x-8y+5}{2}&=\frac{10x^2-12y^2-14xy+2x}{5x+3y+3}+2\\ \sqrt{(6+x)}:\sqrt{(6-y)}&::3:2\end{aligned}\right\}$$

3. $$(x^{\frac{4}{3}}+1)\,(x^{\frac{1}{3}}-1)^2=2\,(x+1)$$

4. $$\left.\begin{aligned}2x+\sqrt{(x^2-y^2)}&=\frac{14}{y}\left\{\sqrt{\frac{x+y}{2}}+\sqrt{\frac{x-y}{2}}\right\}\\ (x+y)^{\frac{3}{2}}+(x-y)^{\frac{3}{2}}&=18\sqrt{2}\end{aligned}\right\}$$

5. There was a run on two bankers, *A* and *B*. After 3 days, *B* stopped payment, by reason of which the daily demand on *A* was tripled, and he failed also after 2 days more. But if *A* and *B* had joined their capitals, they might both have stood the run as it was at first for 7 days, when *B* would have owed £4000 to *A*. What was the daily drain at first on *A*'s bank?

6. The gas-contractors undertake to light a shop with 5 large and 3 small burners; but, having by them only one large burner, supply the deficiency with 5 small ones. The shopkeeper, not finding this light sufficient, procures two more small burners, and agrees for all the lights to burn double the usual time on Saturday nights; and for the additional gas he paid 31*s* *per annum.* What did he pay altogether?

7. A man, who is not aware that his watch gains uniformly, engages to ride from Cambridge to London in 9 hours, and sets his watch by St. Mary's at starting. Upon looking at it after having gone halfway, he supposes it necessary to increase his pace in the ratio of 4 : 3, and consequently reaches London 15′ within the time. But if the watch had lost at the same rate, and he had looked at it at the end of the 14th mile, and then regulated his pace accordingly, he would have been in London too late by 7′. Find the distance from Cambridge to London.

10.

1. $$\frac{x - 1\frac{25}{26}}{2} - \frac{2 - 6x}{13} = x - \frac{5x - \frac{1}{4}(10 - 3x)}{39}$$

2. $$\left.\begin{array}{l} \sqrt{y} - \sqrt{(y - x)} = \sqrt{(20 - x)} \\ \sqrt{(y - x)} : \sqrt{(20 - x)} :: 3 : 2 \end{array}\right\}$$

3. $$8\sqrt{(3x)} + \frac{243 + 324\sqrt{(3x)}}{16x - 3} = 16x + 3$$

4. $$\left.\begin{array}{l} x + \sqrt{(3y^2 - 11 + 2x)} = 7 + 2y - y^2 \\ \sqrt{(3y - x + 7)} = \dfrac{x + y}{x - y} \end{array}\right\}$$

5. From each of two bags, containing different numbers of balls, *A* draws a handful; and now the number in the larger bag is the cube of that in the less, and just the square of one handful. He then draws out of the larger until the number left is the square of that in the less, and now empties the larger into the less, and finds its original number increased by two-thirds. Find the number of balls in each bag at first.

6. *A* and *B* are towns beside a river, which runs at the rate of 4 miles an hour. A waterman rows from *A* to *B* and back again, and takes 39′ more to do it than if there had been no stream. The next day he does the same with another waterman, with

whose help he can row half as fast again: and they are now only 8′ longer than if there had been no stream. At what rate would the waterman row by himself without any stream?

7. Two master bricklayers undertake to lay the foundation of a new court, each taking a part and beginning together. If they had worked together till the whole was finished, it would have taken only $\frac{4}{5}$ of the time it actually took to finish it; and B would have done enough to occupy A three months, and A enough to occupy B twelve months, which is 36 yds more than A actually did. How many yards were there in all?

II.

1. $$\frac{6x - 7\frac{1}{3}}{13 - 2x} + 2x + \frac{1 + 16x}{24} = 4\tfrac{5}{12} - \frac{12\frac{5}{8} - 8x}{3}$$

2. $$\left.\begin{array}{l} x(bc - xy) = y(xy - ac) \\ xy(ay + bx - xy) = abc(x + y - c) \end{array}\right\}$$

3. $$16(x^2 + 2)^{\frac{3}{2}} + \frac{3}{\sqrt{(x^2 + 2)}} = 32x^2 + 48$$

4. $$\left.\begin{array}{l} 30\sqrt{\dfrac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{4}{3}}y^{\frac{4}{3}}}} + 40\sqrt{\dfrac{x^{\frac{2}{3}}y^{\frac{2}{3}}}{x^{\frac{2}{3}} + y^{\frac{2}{3}}}} = 241 \\ \left\{1 + \left(\dfrac{y}{x}\right)^{\frac{2}{3}}\right\} \cdot \left\{3x^{\frac{4}{3}}y^{\frac{2}{3}} + \frac{91}{216}\sqrt{(x^2 + x^{\frac{4}{3}}y^{\frac{2}{3}})}\right\} = \left(\dfrac{5}{6}\right)^3 - x^2 - y^2 \end{array}\right\}$$

5. Two clocks strike the hour, and are heard to strike 19 times. They differ 2″ in time, and one strikes every 3″, the other every 4″. When they strike *together*, it cannot be distinguished whether one or both are striking; and this is the case with the last stroke of the faster clock. What hour did they strike?

6. A revenue cutter observes a smuggler q leagues directly to windward; and gives chase, sailing at $5\frac{1}{2}$ points from the wind, and making tacks of $4p$ miles. The smuggler lies off on the other tack at $2\frac{2}{3}$ points, making tacks of $\frac{1}{3}p\sqrt{3}$ miles, its rate of sailing being to the cutter's as $1 : 4\sqrt{3}$. They sail half the above distances before the first tack. In what tack will the smuggler, while lying in the eye of the wind, first be within range of the cutter's guns, which carry r miles?

7. In the first and least considerable irruption of the Thames into the Tunnel, the water rose in the vertical shaft 8 times as fast as in the horizontal levels in the second. If the levels at the second influx had been 110 feet longer, the velocities of the water ascending in them in the first and second irruptions, and when thus increased would have formed an A.P., the common difference being $\frac{1}{8}$ of the difference of the velocities with which the water rose in the shaft in the two irruptions; and, if the levels had been of the same length on both occasions, the first time of filling would have been half as long again.

The tunnel consisted of two equal levels, terminated by a vertical shaft of twice the breadth of either. The sections of the shaft and levels are supposed to be squares; and the height of the shaft above the upper surface of the levels to be double of its breadth. Given the first time of filling to be 10′ less than the second, find the duration of the latter.

12

1. $\dfrac{7x+6}{28} - \dfrac{2x+4\frac{2}{7}}{23x-6} + \dfrac{x}{4} = \dfrac{11x}{21} - \dfrac{x-3}{42}$

2. $\left.\begin{aligned} \sqrt{(x-y)} + \tfrac{1}{2}\sqrt{(x+y)} &= \frac{x-1}{\sqrt{(x-y)}} \\ x^2+y^2 : xy :: 34 &: 15 \end{aligned}\right\}$

3. $\dfrac{x^{(m-n)^2} + x^{-4mn}}{x^{(m-n)^2} - x^{-4mn}} = a^{\frac{r}{s}}$

4. $\left.\begin{aligned} 5 - 2\sqrt{(y+2)} &= \tfrac{9}{64}x^2 - (\sqrt{x} - 3\sqrt{y})^2 \\ \frac{7}{y} - 10\sqrt{\frac{x}{y}} &= x - 16 \end{aligned}\right\}$

5. A question was lost on which 600 persons had voted. The same persons having voted again on the same question, it was carried by twice as many as it was before lost by, and the new majority was to the former :: 8 : 7. How many changed their minds?

6. Three towns *A*, *B*, *C*, lie at the angles of a right-angled triangle, *B* at the right angle, and the distance *AB* being the least of the three. A pedestrian finds that the time of his going from *A* to *B*, and then from *B* to *C*, exceeds the time from *A* to *C*

direct by $2\frac{2}{3}$ hours. A coach, which left A four hours after him, and travels thrice as fast, overtakes him 8 miles from B in the way to C; and after passing through C to A, and waiting there $6\frac{2}{3}$ hours, it makes the same circuit, and reaches A again at the same time with the pedestrian, who had rested four hours at C. Find the rate at which he walks.

7. A, B, C, D, are rough diamonds. The value of C in £'s is less by 52 than the weight of A in carats, and the value of C and D in £'s is equal to the weight of B in carats. Each loses half its weight by cutting: but the dust from A and B is worth £85; and the value of A : that of C, D, and the dust from A :: half that of B : that of the dust from B. A diamond, weighing one carat when rough, is worth £3 when cut, and £2 when uncut: the value $\propto$ the square of the weight, and the dust is worth £1 per carat. Find the value of D when cut.

13.

1. $$\frac{6-5x}{15} - \frac{7-2x^2}{14(x-1)} = \frac{1+3x}{21} - \frac{2x-2\frac{1}{5}}{6} + \frac{1}{105}.$$

2. $$\left.\begin{aligned} 3x + \tfrac{2}{3}\sqrt{(xy^3+9x^2y)} &= (x-\tfrac{1}{3})\,y \\ 6x+y : y :: x+5 &: 3 \end{aligned}\right\}$$

3. $$\sqrt{x} - \frac{8}{x} = \frac{7}{\sqrt{x}-2}$$

4. $$\left.\begin{aligned} x^3y - 4 &= 4x^{\frac{1}{2}}y - \tfrac{1}{4}y^3 \\ x^{\frac{3}{2}} - 3 &= x^{\frac{1}{2}}y^{\frac{1}{2}}\left(x^{\frac{1}{2}} - y^{\frac{1}{2}}\right) \end{aligned}\right\}$$

5. A and B start for a race which lasts 6′. At the end of 4′, the distance between them is $\frac{1}{440}$ of the length of the course. After 1′ more, B, who is last, quickens his horse 20 yds a minute, and comes in 2 yds before A, who has gone uniformly throughout. Find the length of the course.

6. The revenue of a state was increased for war in the ratio of $2\frac{1}{4} : 1$; and, after deducting the expense of collecting and the interest of the national debt, the net income was augmented in the ratio of $3\frac{12}{23} : 1$. If, however, the revenue had been reduced in the ratio of $1\frac{7}{9} : 1$, the net income would have been diminished in the ratio of $7\frac{2}{3} : 1$, and would in fact have just amounted to 4 millions. Find the revenue before the increase, supposing the

expense of collecting to vary as the square root of the sum collected.

7. Three boats, A, B, and C, start in a race, B being 20 yards behind A, and C the same behind B. A and C set off at an uniform rate, C making a yard less *per* stroke than A. But B took 7 strokes to 6 of A or C, and increased its speed besides by 3 in. every stroke; so that, when A had taken 42 strokes, B, which had lost 16 yards by steering, was only a yard behind A. At this point B's speed decreased twice as fast as it had increased before; while C, quickening its strokes at the same instant in the ratio of 6 : 7, and gaining each stroke as much speed as B lost, at the end of 28 strokes overtook B, which had lost 11 yds more by steering. Compare the velocities with which they started.

14.

1. $$\left.\begin{aligned} x - \frac{2y - x}{23 - x} &= 20 - \frac{59 - 2x}{2} \\ y + \frac{y - 3}{x - 18} &= 30 - \frac{73 - 3y}{3} \end{aligned}\right\}$$

2. $$\{a^2 - a\sqrt{(b^2 + bx)} + \tfrac{1}{2}bx - \tfrac{1}{4}b^2 + \tfrac{1}{2}b\sqrt{(x^2 - bx + b^2)}\} \times$$
$$\{a^2 + a\sqrt{(b^2 + bx)} + \tfrac{1}{2}bx - \tfrac{1}{4}b^2 - \tfrac{1}{2}b\sqrt{(x^2 - bx + b^2)}\} = a^4 - \tfrac{3}{16}b^4$$

3. $$2x\sqrt{(1 - x^4)} = a(1 + x^4)$$

4. $$\left.\begin{aligned} (2 + 4xy - 3x^2)^2 &= 2 - 4x^2y^2 + 3x^4 \\ (x^2 - 1)^2 &= (2y^2 + x^2 + 1)(2y^2 - x^2 - 1) \end{aligned}\right\}$$

5. The owner of a balloon calculated that, if he filled the enclosure, which he had hired for the day at £5, with spectators at 2*s* each, and two persons ascended with him, he should make a profit of 140 per cent on his outlay. But, the gas and the weather proving bad, he pays but half the price of inflating, and ascends alone with the enclosure a fourth part full, losing on the whole $\frac{1}{3}$ of his outlay. On the next day he ascends again with a full balloon, the enclosure $\frac{3}{4}$ filled, and one companion, and by the whole speculation gained £10. What was the cost of inflation?

6. A and B row between two places, B in a certain time by his watch: but A, when he has gone by his watch the same time, relaxes his speed, and moves only $\frac{2}{3}$ as fast as before. Now, if this take place in going down the stream, the first part of the distance will take A six times as long as the last, but, if in going up, only

the same time; and this would also be the case in going upwards, if *A*, instead of relaxing, were even to increase his speed in the ratio of 7 : 5, provided that he exchange watches with *B* at starting. Supposing their watches to gain uniformly, compare the rates of rowing of *A* and *B*.

7. A regiment, in which there are between 10 and 100 officers and twice as many serjeants, in clearing the streets during a revolution, loses 2 officers; and, after storming a barricade in which three more fall, is obliged to retreat, taking in a volunteer as officer, but, in so doing, loses other three. While clearing the streets, the liability of an officer to fall is half that of a serjeant or private; but at the barricade as 4 : 3, and in the retreat as 3 : 4. Also, on their leaving the barracks, the number, whose two left-hand digits express the number of serjeants and its other digits that of officers, exceeds by 20 ten times the number of privates; but, on their return, (having parted with the volunteer,) it exceeds it only by 12, the number of officers being still above 10. Find the state of the regiment at first.

15.

1. $\dfrac{x^2+1}{4x^2-1} = \dfrac{x}{1+2x} - \dfrac{1}{4}$

2. $\left.\begin{array}{l}\sqrt{x} - \sqrt{y} = \sqrt{x}\,(\sqrt{x} + \sqrt{y}) \\ (x+y)^2 = 2\,(x-y)^2\end{array}\right\}$

3. $\sqrt{x}\,\sqrt{(x^3 + a^3)} = x^2 - a^2 + 3ax$

4. $\left.\begin{array}{l}\sqrt[3]{\dfrac{27y^{\frac{3}{2}} - 1}{x^3 + 3y^2 - 2xy^{\frac{3}{2}}}} = 3\sqrt{\dfrac{x}{y}} \\ 3x^2 + 42xy + 16y^2 = 4\sqrt{(xy)}\,(5x + 11y)\end{array}\right\}$

5. A farm was rated at 3*s* an acre, and the tenant, on receiving back 10 per cent of his rent, found that the sum returned was £6 more than the whole rate. The next year the rates were doubled, and he received back 15 per cent of his rent; but now the sum returned only just paid for the rate. What was his rent?

6. *A* starts to walk from Cambridge to London at the rate of $3\frac{1}{2}$ miles per hour. In $2\frac{1}{2}$ hrs the Times passes him, and the Fly at 10′ to 10; he rests $2\frac{1}{2}$ hrs on the road, and again meets the Times on its return, and half-a-mile farther the Fly, at 20′ after 5. The Times left Cambridge at 6, and the Fly at $\frac{1}{2}$-past 7, and both started from London at 3: find the distance from Cambridge to London.

7. At an election of one member to Parliament, one-third of the electors gave plumpers for C, and those given for A and B were $\frac{4}{21}$ of the whole number of votes given. Of those electors who gave single votes to C, twice as many voted for B as for A: and B stood at the head of the poll with a majority of 110 over C. A scrutiny being demanded, it appeared that those who had split their votes between A and B had no legal right to vote, and C is now returned with a majority of 200 over A. Find the final state of the poll, it being observed that A has now as many single votes as plumpers.

16.

1. $$\frac{3x-2}{4} + \frac{x}{2} - 11\tfrac{5}{6} = \frac{x - \frac{1}{3}(4x-9)}{6} - 5$$

2. $$\left.\begin{aligned} 3y + 11 &= \frac{4x^2 - y(x+3y)}{x-y+4} + 31 - 4x \\ (x+y)(y-2) + 3 &= 2xy - (y-1)(x+1) \end{aligned}\right\}$$

3. $$x - 2\sqrt{(x+2)} = 1 + \sqrt[4]{(x^2 - 3x + 2)}$$

4. $$\left.\begin{aligned} (1-x^2)^2(1+y^2) - (1+x^2)^2(1-y^2) &= 4x^2\sqrt{(1+y^4)} \\ 4xy &= \sqrt{2}(1-x^2)(1-y^2) \end{aligned}\right\}$$

5. A and B engaged to reap equal quantities of wheat, and A began half-an-hour before B. They stopped at noon to rest an hour, and observed that just half the work was done. B's part was finished at 7 o'clock, and A's at $\frac{1}{4}$ to 10. At what hour did A begin?

6. Two boys start from the right angle of a triangular field, and run along the sides with velocities in the ratio of 13 : 11. They meet first in the middle of the opposite side, and again 30 yards from the starting point. Find the length round the field.

7. A stable-keeper bought two horses for £50, and sold them, one for double, and the other for half, of what he gave for it. The former had produced for its hire only half of what it cost him, and cost in keep as much per cent on its price as the hire of the other produced on its price, the latter being kept for $\frac{4}{5}$ as many guineas as it sold for pounds. The keep of the two cost £33, and he made by them, upon the whole transaction, nine times his profit on the sale. What did each cost?

17.

1. $\dfrac{1}{3} - \dfrac{7x-1}{6\frac{1}{2} - 3x} = \dfrac{8}{3} \cdot \dfrac{x - \frac{1}{2}}{x - 2}$

2. $\left.\begin{aligned} x^m a^n + y^n b^m &= 2\,(ax)^{\frac{m}{2}} \,.\, (by)^{\frac{n}{2}} \\ xy &= ab \end{aligned}\right\}$

3. $x^2 - 2x + 4 = 2\sqrt{(x^2 - 1)}$

4. $a^2\,(x^4 + y^4) = y^6 - 2a^2 xy\,\sqrt{(x^4 - y^4)}$
$a^2\,(x^4 - y^4) = x^2 y^2\,(2a^2 - x^2)$

5. *A* and *B* run a race to a post and back again. *A* returning meets *B* 90 yards from the post, and reaches the starting-place 3′ before him. If he had then returned, he would have met *B* at a distance from the starting-place equal to $\frac{1}{6}$ of the whole distance. Find the length of the course, and duration of the race.

6. The upper spokes *R* and *r* of the hind and fore wheels of a carriage are vertical at starting. When *r* has revolved once, it is at right angles to the spoke next before *R*; and when *R* has made $\frac{2}{3}$ of a revolution, *r* ascending again makes the same angle with an horizontal line as the spoke next before it. Given diam. of forewheel : diff. of heights of axles : : number of spokes in forewheel : 2, find the number of spokes in each.

7. *A* agrees with his steward to allow a per centage on the rents he collected, on condition of his returning half the same on the rents not paid. The first year the steward's income amounts to 6 per cent on the whole rental; but in the next, in order to obtain the same income, he makes a return of rents received £270 under their value. In the third year, though the rents are reduced $7\frac{1}{2}$ per cent, the amount not paid is the same as in the second year; the steward's income is only $\frac{3}{4}$ of his first year's, and, to make it up, he doubles his last year's fraud. Required the rental of the estate.

18.

1. $\dfrac{1}{2}\left(\dfrac{2}{3}x + 4\right) - \dfrac{7\frac{1}{2} - x}{3} = \dfrac{x}{2}\left(\dfrac{6}{x} - 1\right)$

2. $\left.\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{4}y + \tfrac{1}{5}z &= 1\tfrac{1}{3} \\ \tfrac{1}{3}y + \tfrac{1}{4}x + \tfrac{1}{10}z &= \tfrac{17}{18} \\ \tfrac{1}{8}z + \tfrac{1}{5}x + \tfrac{1}{6}y &= \tfrac{169}{180} \end{aligned}\right\}$

3. $\dfrac{1 + x^5}{(1 + x)^5} = a$

4. $\left.\begin{aligned} y + 3\sqrt[3]{y}\,\{\sqrt[3]{(a + bx)} - \sqrt[3]{y}\}\,\sqrt[3]{(a + bx)} &= 2a \\ \dfrac{y - 3\sqrt[3]{y}\,\sqrt[3]{(a^2 - b^2x^2)}}{\sqrt[3]{y} - \sqrt[3]{(a + bx)}} &= 2a\,\sqrt[3]{(a - bx)} \end{aligned}\right\}$

5. Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he did for two-thirds of the time, in which Silenus would have emptied the cask. After that Silenus wakes, and drinks what the other had left. Had they both drunk together, the cask would have been emptied two hours sooner, and Bacchus would have drunk only half of what he left for Silenus. Find the time in which each by himself would have emptied the cask.

6. *A*, *B*, *C*, whose powers are in the ratio of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, engage to reap a field of wheat at so much per acre, to be proportioned among them according to their work. After two days a quarrel arises and *C* withdraws, receiving for his labour 5*s*, the farmer having made a deduction from the stipulated price per acre, on account of the delay thus occasioned. *A* and *B* then engage to finish the work on the same terms as at first, which takes them another day. *A* receives upon the whole 1*s* 6*d* less than he would have done if *C* had not withdrawn; while the farmer saves in money half as much as he agreed to pay per acre. Find the number of acres.

7. *P*, *Q*, *R*, represent three candidates at an election. *Q* polled as many plumpers wanting one as the split votes between *P* and *R* exceeded those between himself and *R*; and the number of split votes between *Q* and *R* was one more than twice the number between *Q* and *P*. If *P* had not voted for himself and *R*, but for *R* only, and if five others who split betwixt *P* and *Q* had voted for *Q* only, *Q* would have just beaten *P*, and would have been 48 below *R*. The number of voters was 1341, of which 565 gave plumpers. Find the final state of the poll.

19.

1. $$\frac{7x-13\frac{1}{2}}{11}-\frac{2}{3}\cdot\frac{x-15}{7}=\frac{15}{14}(x-1)$$

2. $$\left.\begin{array}{l}\dfrac{2x}{x+y}-\dfrac{y^2}{x^2-y^2}=\dfrac{13x+16y}{8(x+2y)}\\ \sqrt{x}+\sqrt{y}=\sqrt{(x+y+\sqrt{3})}\end{array}\right\}$$

3. $$\sqrt{\{2(x+2)\}}-2\sqrt{(2-x)}=\frac{12x-8}{\sqrt{(9x^2+16)}}$$

4. $$\left.\begin{array}{l}ax^2+by^2=a(x+b)\\ x^4+x^2(y^2-2x)-x(y^2-1)+y^4=a^2\end{array}\right\}$$

5. In a race between two boats a spectator, walking at the rate of 5 miles an hour, is $\frac{1}{8}$ of a mile a-head of the first boat at starting; and, when it passes him, he observes that the interval between the boats, which at first was 30 yards, is reduced to 20. At $1\frac{1}{4}$ mile from where it started, the first boat is overtaken by the second: how long did the race last?

6. When wax candles are 2*s* 6*d* per lb, a composition is invented, such that a candle made of it will burn $\frac{2}{3}$ of the time in which a wax candle, of the same thickness and $\frac{1}{4}$ as heavy again, would burn. If the two give an equally bright light, what must be charged per lb for the composition, that it may be as *cheap* as wax?

7. Into a cubical cistern, 8 feet deep, and having an unknown leak, water is poured from two pumps worked by A and B. They pump together till it is half filled, when B falls asleep, but A goes on pumping till it is three-fourths filled, and then goes away. B upon waking finds it still half full, and, after pumping till it is again three-fourths filled, departs also to look for A. They return together, and find the water $1\frac{1}{2}$ in. lower than when B left. The leak is now discovered and stopped; and the vessel is filled by them in half the time in which they had worked together at first. Now $10\frac{1}{3}$ hours had elapsed since they first began pumping, and B had worked alone twice as long as A had. Given that a cubic foot contains $15\frac{6}{8}$ gals, find the quantity of water thrown in per hour by each pump.

20.

1. $\dfrac{4x-17}{9} - \dfrac{3\frac{2}{3}-22x}{33} = x - \dfrac{6}{x}\left(1-\dfrac{x^2}{54}\right)$

2. $x(y+z)^2 = 1 + a^3$, $x + y = \frac{3}{2} + z$, $yz = \frac{3}{16}$

3. $4\{(x^2-16)^{\frac{3}{4}} + 8\} = x^2 + 16(x^2-16)^{\frac{1}{4}}$

4. $(x^6+1)y = (y^2+1)x^3$, $(y^6+1)x = 9(x^2+1)y^3$

5. A at his death leaves a certain property in money and sums due to him. The executors invest the money in the funds at 96. Of the debts $\frac{1}{6}$ is not recovered; and, when the stock is sold out at 92, the heir (it is found) receives less by £140 than he would have done, if the debts had been completely recovered. His loss is also $\frac{1}{8}$ of the sum he receives. Find the amount of the debts and money.

6. A ferry-boat was about to cross a river, when it was upset by a party leaping in, who increased the n° of persons in the boat in the ratio of 4 : 5. The n° who got out without help (including the ferryman) was $\frac{5}{6}$ of the increased n° of passengers, and the n° helped out was $\frac{1}{3}$ of the n° of minutes the last man was in the water. The numbers extricated in both ways in each of the first three minutes form a series of fractions, whose numrs increase in A.P., and denrs in G.P., the common ratio and difference being the same as the n° of persons in the water at the end of the three minutes, and the first numr (with denr *unity*) being the n° of minutes remaining till the last man was out. Had, however, the n° of intruders been less by four, and still increased the n° of persons in the boat in the same ratio, the increased n° of passengers would have been the same as twice the common ratio of the G. P. How long was the last man in the water?

7. Towards the end of a cricket match, the second party were a certain n° of notches behind, and had still three men, A, B, C, remaining. A and B are in, and after $\frac{5}{8}$ of the n° have been gained, A is struck out, and C takes his place. Now B scores as many notches in C's innings, as there were bye-balls in A's, and as many in A's as were gained altogether in C's. If also the byes in A's innings be added to B's notches in it, and the byes in C's innings to C's notches, these quantities will be inversely proportional to the corresponding nos of byes. C gets one more notch than B in their common innings, and the party loses by 3: but, if B's scoring be reversed, that is, if he be supposed to get as many notches in A's innings as the n° of byes in C's, and as many in C's as the whole n° now gained in A's, the three would have scored between them, (not reckoning the bye-balls,) just as many as their whole former number. How many notches did A score?

21.

1. $\dfrac{3x}{2} + \dfrac{81x^2 - 9}{(3x-1)(x+3)} = 3x - \dfrac{3}{2} \cdot \dfrac{2x^2 - 1}{x+3} - \dfrac{57 - 3x}{2}$

2. $xy + z = 5,\ \ xyz = 4,\ \ 2(x^2 - y) = (y^2 - x)^2$

3. $(x+3)^2 - 2(x^2 + 3) = 2x(x+1)^2$

4. $\left.\begin{aligned} (x+y)^2 &= x^4 + x^2y^2 + y^4 \\ x^4 + 4y^4 &= 4xy(2y^2 - x^2) \end{aligned}\right\}$

5. In a tithe commutation, the rent-charge was paid at 3*s* an acre, and the tithe-owner found the first year that his rates wanted £6 of being 10 per cent on his receipts. The next year the rates were doubled, and amounted to 15 per cent on his receipts. What was the number of acres in the parish?

6. An omnibus starts with a certain n° of passengers, and takes up four more on the road, whose fare is the same as that paid by the others. On deducting $\frac{1}{12}$ of the whole fare for expences, there remains a profit of 4*s* 7*d*. But if those who were last taken in had paid half as many pence as there were passengers altogether, the money received would have exceeded by 2*s* 8*d* double the difference of the sums actually paid by the two sets of passengers. With how many did the omnibus start?

7. *A* and *B*, having a single horse, travel between two milestones distant an even n° of miles in $2\frac{62}{63}$ hrs, riding alternately mile and mile, and each leaving the horse tied to a milestone until the other comes up. The horse's rate is twice that of *B*: *B* rides first, and they come together to the 7th milestone. Finding it necessary to increase their speed, each man after this walks half-a-mile an hour faster than before, and the horse's rate is now twice that of *A*, *B* again riding first. Find the distance they travelled.

22.

1. $$\frac{1}{x^2+11x-8}+\frac{1}{x^2+2x-8}+\frac{1}{x^2-13x-8}=0$$

2. $(x-2)^2+(y-3)^2+(z-1)^2=24,\quad xy+yz+xz=63,\quad 2x+3y+z=30$

3. $$\frac{(n-1)\,(a^4+a^2x^2+x^4)}{(n+1)\,(a^4-a^2x^2+x^4)}=\left(2-\frac{1}{n}\right)\left(\frac{ax}{a^2-x^2}\right)^2$$

4. $$\left.\begin{array}{c}(x^4+2bx^2y+a^2y^2)\,(y^4+2bxy^2+a^2x^2)=4\,(a^2-b^2)\,(b+c)^2x^3y^3\\ x^3+y^3=2cxy\end{array}\right\}$$

5. A merchant, travelling from St. Petersburg to Moscow, had provided himself with notes of the Bank of Russia, amounting in all to 540 roubles. The paper at first bore the value marked on it; but south of Torjok, a town on the road, and in Moscow itself, a premium of 20 per cent was allowed on each note. On reaching Moscow, he received 432 roubles for the notes that remained: he spent there 237 roubles, and had just enough left to pay his expences back, supposing them the same as before. How many roubles did he spend between St. Petersburg and Moscow?

6. From a quantity of gold, silver, and copper, weighing in all 20300 oz., two alloys were formed. In the one gold and copper were mixed in the proportion of 11 : 1, in the other silver and copper, in the proportion of 37 : 3; and there were 288 oz. of copper over. The alloy of gold and copper was coined at the rate of £3 17*s* 10½*d* per oz., and that of silver and copper at the rate of 5*s* 6*d* per oz. The whole sum thus produced was £5546 14*s* 6*d*. How many ounces were there of each metal?

7. A body of 6048 soldiers was divided into a number of equal detachments, and sent to occupy as many fortresses. In the course of the campaign as many as two whole garrisons and half of another died of an epidemic, and all the rest except 84 invalids, who returned to head-quarters, were equally divided among the fortresses as before. But, the reduced garrisons proving too weak for their defence, all the fortresses fell into the hands of the enemy, and the men, with the exception of four whole garrisons and 210 fugitives, were killed or made prisoners. The loss thus sustained, together with that caused by the epidemic, amounted to 4186 men. Required the number of fortresses.

23.

1. $$\frac{x-4\frac{2}{3}}{3}-\frac{2x-3\frac{2}{3}}{4}=\frac{3}{2}\left\{x-\frac{x-1\frac{1}{2}}{2}\right\}+\frac{4x}{3}\left\{x-3-\frac{(x-1)(x-2)}{x}\right\}$$

2. $$\left.\begin{aligned}&\frac{2}{3}\left\{x-\frac{3}{5}y\right\}+\frac{x+\frac{1}{5}y}{6}=\frac{1}{3}-\frac{1}{2}\left\{\frac{\frac{4}{5}y-2}{6}-(x-y)\right.\\ &x-2y-\frac{3y-5x}{2}=\frac{11}{2}(x+y)+3(x-y)\left.\vphantom{\frac{1}{2}}\right|\end{aligned}\right\}$$

3. $$\left(x-\tfrac{1}{3}\right)^2-\tfrac{25}{9}=\frac{3x^2+\frac{4}{9}}{2\left(x-\frac{1}{3}\right)+\sqrt{\left\{x\left(x-\frac{8}{3}\right)\right\}}}$$

4. $$\left.\begin{aligned}&\frac{3+2x^2-4x^4}{x^2-1}=y^2(1-2y^2)\\ &(2x^2-1)(2y^2-1)=3\end{aligned}\right\}$$

5. A person leaves London for Derby by the Birmingham Railway at 10 A.M., intending to get upon the Midland Counties line at Rugby, and allowing for a delay of 30′ in changing trains, but expecting to travel the 48 miles from thence to Derby at the same rate at which he had come down, he calculated to reach Derby at 4 P.M. On reaching Rugby, however, he finds that there will be no train for Derby till too late for his purpose; but that by going

on upon the first line to Hampton ($\frac{1}{4}$ as far again as he had come already) he might start immediately by the Derby Junction line, and though the whole distance by this route would be 13 miles longer than by the other, yet the speed on the second line being one mile an hour quicker, he would reach Derby just $1\frac{1}{2}'$ before 4. What is the distance from London to Rugby?

6. Two cubical boxes A, B, of which A is larger by 1216 cubic inches, are filled with balls, there being 12 more along an edge of B than in an edge of A, and the number of balls in the faces of A being to the number in the edges of B as 7 : 22. Also the difference between the areas enclosed by the balls of B, (defined by a thread passing round them,) when they are spread out first into a hollow and then into a solid square, is to the same difference with respect to the balls of A as $129\frac{181}{900}$: 1. Find the radii of the balls.

7. The income of a schoolmaster arises partly from ten pupils residing in his house; and partly from an endowment, which produces a certain number of quarters of wheat each year. When wheat sells for 60*s* the expenditure of his family (£249) is less than his savings by a number, which when divided by twice the number of his pupils expresses the proportion which the clear gain bears to the whole charge for each pupil. In the following year wheat falls to 55*s*, and a tax of 8*d* in the pound is laid upon income, payable upon the net income of the preceding year; but the cost of living for his pupils being diminished, (so that, in fact, the amount of income-tax he has to pay, with 10*s* added, would just support one pupil,) he finds that his savings are greater than in the year previous by a sum equal to the difference of his net income in the two years, which is $\frac{1}{18}$th of the expenditure of his family in the second year, besides allowing for an outlay of £15 in repairs. The net income from pupils in the first year being £330, find that from the endowment in the same year and the ratio of the costs of living in the two years.

24.

1. $\sqrt{\{(x-1)(x-2)\}} + \sqrt{\{(x-3)(x-4)\}} = \sqrt{2}$
2. $x(\sqrt{x}+1)^2 = 102(x+\sqrt{x}) - 2576$
3. $\left(\dfrac{2x+3}{2x-3}\right)^{\frac{1}{3}} + \left(\dfrac{2x-3}{2x+3}\right)^{\frac{1}{3}} = \dfrac{8}{13} \cdot \dfrac{4x^2+9}{4x^2-9}$
4. $a^2 - x^2 = 3xy, \quad (\sqrt{y}-\sqrt{x})(a-x) = 3\sqrt{x}(x+y)$

5. *A* and *B* embark in trade for 5 years; *B* is to have $\frac{7}{16}$ of the net annual profits for the first half of the time, and half of them for the remainder. After $3\frac{1}{2}$ years the annual profits, by a lowering of the tariff, were increased in the proportion of 6 to 5, and at the same time became liable to a reduction of 7 pence in the pound by the laying on of the Income-tax. At the termination of the partnership, *B*'s share of the total net profits amounted to £987. Required the annual profits of the business before the duties were reduced.

6. A cubical vessel is filled with water and has its surface exposed to the sky. The temperature of the atmosphere is 30° on the first day, and every successive day it is increased by 1°. Suppose that a temperature of 15° would evaporate 1 inch in one day, and other temperatures in the same proportion. Every evening there are showers; on the first evening 3 inches of rain fall, and the depth of rain falling each successive evening decreases in an A.P. whose common difference is $\frac{1}{90}$th of what fell on the first day. At the end of 41 days the vessel is found to be empty; required its content.

7. *A* and *B* set out together, and walk from Keswick over Helvellyn to Ambleside. *B* arrives at the foot of Helvellyn, which is 5 miles from Keswick, in an hour, and then slackens his pace so that he just reaches the top at the end of the second hour. Here he sits down to rest, until *A* passes him and gets as much before him (*B*) as he was behind when *B* first sat down. *B* then starts at an increased pace, and passes *A* at the end of the third hour from the time of starting. *B* walks on at the same pace for another hour, and then waits for *A* to come up, whose distance behind him was $\frac{2}{3}$ of what it was at the end of the first hour. When *A* overtakes him, he starts again and walks at the same pace until he reaches Ambleside; the time of this last stage being to that of his first rest in the ratio of 20 : 7. On his arrival, he fires a pistol for the information of *A* who, having hitherto kept up a uniform pace without stopping, now diminishes it in the ratio of 5 : 7, and reaches Ambleside 10 minutes after *B*. What is the distance from Keswick to Ambleside by this route?

25.

1. $$\frac{\frac{3}{x}-1}{2}-\frac{9\left(\frac{1}{2x}-1\right)-\frac{2}{5}\left(\frac{9}{2x}-4\right)}{\frac{3}{x}-4}=\frac{\frac{9}{x}+19}{6}$$

2. $\sqrt{(a+x)}+\sqrt{(a-x)}=\sqrt{\left(\frac{3b^2+x^2}{a+b}\right)}$

3. $\left.\begin{array}{l} x^3 = 31x^2 - 4y^2 \\ y^3 = 31y^2 - 4x^2 \end{array}\right\}$

4. $(x+y)^{\frac{1}{3}}+(x-y)^{\frac{1}{3}}=a^{\frac{1}{3}}$, $(x^2+y^2)^{\frac{1}{3}}+(x^2-y^2)^{\frac{1}{3}}=a^{\frac{2}{3}}$

5. A railway train travels from *A* to *C* passing through *B* where it stops 7 minutes. Two minutes after leaving *B*, it meets an express train which started from *C* when the former was 28 miles on the other side of *B*. The express travels at double the rate of the other, and performs the journey from *C* to *B* in $1\frac{1}{2}$ hours; and, if on reaching *A* it returned at once to *C*, it would arrive 3 minutes after the first train. Find the distances between *A*, *B*, and *C*, and the speed of each train.

6. To meet a deficiency of (m) millions in the revenue, an *additional* tax of a per cent. was laid on exports, and the tax on imports was diminished (c) per cent.: the value of the imports was in consequence increased so as to be n times as great as the exports, and the deficiency was made up. Now, if the additional tax on exports had been a' per cent, and the tax on imports diminished c' per cent, the values of the articles being altered as before, the deficiency would not have been made up by m' millions. Find the exports and imports after the alteration.

7. Fifty thousand voters, who have to return a member to an assembly, are divided into sections of equal size, and each section chooses an elector, the member being returned by the majority of such electors. There are two candidates, *A* and *B*. In those sections which return electors favourable to *A*, the majority is double the minority, while, in those favourable to *B*, the minority forms only a tenth of the whole. After the primary elections *C* comes forward, and is returned by a majority of 3 over *A*, and 14 over *B*. If *C* had not come forward, *A* would have been returned by a majority 19 less than the whole number of *C*'s votes, and if the 50,000 had voted *directly* between *A* and *B*, *B* would have had a majority of 6000. Find the number of sections.

ANSWERS TO THE EXAMPLES.

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1. 1. $-27,\ 1,\ -27y^3 - 27y^2 - 9y - 1$.

4. $a^5 + a^4x + a^3x^2 + a^2x^3 + ax^4 + x^5 + \dfrac{2x^6}{a-x},\quad 5x + 1 + \dfrac{2}{5x-1},$

$2x - 3 + \dfrac{18}{2x+3},\quad 4m^2 + n^2 + \dfrac{2n^4}{4m^2 - n^2}$.

5. $3mn - 2 + \dfrac{8}{3mn+2},\ 1 + 2x + 4x^2 + \dfrac{16x^3}{1-2x},\ 9x^2 + 3x + 1 + \dfrac{2}{3x-1}$.

6. $1 + 2x + 4x^2 + 8x^3 + \dfrac{32x^4}{1-2x},\ a^3b^3 + 2a^2b^2 + 4ab + 8 + \dfrac{32}{ab-2},$

$a^4 - 2a^3b + 4a^2b^2 - 8ab^3 + 16b^4 - \dfrac{64b^5}{a+2b}$.

7. $a + b - 2c + \dfrac{8c^3}{a+b+2c},\ (x+y)^2 + (x+y)z + z^2 + \dfrac{2z^3}{x+y-z}$.

8. $(x+y)^2 - 2(x+y)z + 4z^2 - \dfrac{16z^3}{x+y+2z},$

$4(x+y)^2 - 2(x+y)z + z^2 - \dfrac{2z^3}{2(x+y)+z}$.

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2. 1. $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \&c,\ 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \&c.$

2. $a - \dfrac{1}{2}\dfrac{b}{a} - \dfrac{1}{8}\dfrac{b^2}{a^3} - \dfrac{1}{16}\dfrac{b^3}{a^5} - \&c,\ a^{\frac{2}{3}} - \frac{1}{3}a^{-\frac{4}{3}}b - \frac{1}{9}a^{-\frac{10}{3}}b^2 - \frac{5}{81}a^{-\frac{16}{3}}b^3 - \&c.$

3. $a + \frac{1}{2}a - \frac{1}{8}a + \frac{1}{16}a - \&c = a(1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \&c),\ a^{\frac{2}{3}}(1 + \frac{1}{3} - \frac{1}{9} + \frac{5}{81} - \&c).$

4. $1 - x + x^2 - x^3 + \&c,\ 1 - \frac{2}{3}x - \frac{1}{9}x^2 - \frac{4}{81}x^3 - \&c.$

2.* 1. $x - p$. 2. $x + a$. 3. $x^2 - x + 1$. 4. $x^2 + y$.
5. $ax - b$. 6. $ax^2 + bx + c$. 7. $x - p$. 8. $x - a$.

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3. 1. $4,\ \frac{2}{3},\ 1,\ 2$. 2. $-3,\ -6,\ \frac{3}{5},\ -\frac{3}{8}$. 3. $2\frac{1}{2},\ \frac{1}{3},\ 2\frac{1}{4},\ -1\frac{1}{8}$.
4. $5\frac{1}{4},\ -\frac{4}{5},\ -1\frac{2}{3}$. 5. $1\frac{5}{9},\ -5\frac{5}{6},\ \frac{81}{84},\ 4\frac{6}{25}$. 6. $3,\ -3\frac{2}{5},\ -7,\ -\frac{1}{8}$.

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5. 1. 2. 2. 3. 3. $1\frac{1}{4}$. 4. $1\frac{1}{3}$. 5. $\frac{2}{3}$. 6. 3.
7. $-\dfrac{4}{5}$. 8. $-2\frac{2}{3}$. 9. $\dfrac{10}{7a}$. 10. $\dfrac{25}{31}$. 11. $-\dfrac{1}{2}$. 12. $\dfrac{a}{b}$.

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6. 1. $\dfrac{4 + 3\sqrt{2} - 2\sqrt{3} - \sqrt{6}}{4}$. 2. $\dfrac{2\sqrt{3} + 3\sqrt{2} - \sqrt{30}}{12}$.

3. $\dfrac{\sqrt{2} + \sqrt{6}}{2}$. 4. $1 + \sqrt{3} + \sqrt{5}$. 5. $\sqrt[3]{4} - \sqrt[3]{6} + \sqrt[3]{9};\ 5$.

6. $a^{\frac{5}{2}} - a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} - ab + a^{\frac{1}{2}}b^{\frac{4}{3}} - b^{\frac{5}{3}};\ a^3 - b^2$.

7. $3^{\frac{5}{2}} + 3^2 . 5^{\frac{2}{3}} + 3^{\frac{3}{2}} . 5^{\frac{4}{3}} + 3 . 5^2 + 3^{\frac{1}{2}} . 5^{\frac{8}{3}} + 5^{\frac{10}{3}}$

$= 9\sqrt{3} + 9\sqrt[3]{25} + 15\sqrt[6]{675} + 75 + 25\sqrt[6]{16875} + 125\sqrt[3]{5}; \ -598.$

8. $a^{\frac{3}{2}} - ax^{\frac{3}{4}} + a^{\frac{1}{2}}x^{\frac{3}{2}} - x^{\frac{9}{4}}; \ a^2 - x^3.$ 9. $x^2 + xy^{\frac{2}{3}} + y^{\frac{4}{3}}; \ x^3 - y^2.$

10. $x^{\frac{3}{2}} - xy^{\frac{1}{4}} + x^{\frac{1}{2}}y^{\frac{1}{2}} - y^{\frac{3}{4}}; \ x^2 - y.$

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7. 1. $x + \sqrt{(a^2 - x^2)}.$ 2. $\sqrt{\{\frac{1}{2}(1 + m)\}} + \sqrt{\{\frac{1}{2}(1 - m)\}}.$
3. $\sqrt{(a + b)} + \sqrt{(a - b)}.$ 4. $m - \sqrt{(mn - m^2)}.$
5. $\sqrt{(a + b)} + \sqrt{c}.$ 6. $1 - x + \sqrt{(1 + 2x - x^2)}.$

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8. 1. $\sqrt[4]{8} - \sqrt[4]{2}.$ 2. $\sqrt[4]{24} + \sqrt[4]{6}.$ 3. $\sqrt[4]{75} - \sqrt[4]{27}.$
4. $\sqrt[4]{2} + \sqrt[4]{\frac{1}{2}}.$ 5. $\sqrt[4]{12} + \sqrt[4]{3}.$ 6. $\sqrt[4]{36\frac{3}{4}} - \sqrt[4]{\frac{3}{4}}.$
7. $\sqrt[4]{20} + \sqrt[4]{5}.$ 8. $\sqrt[4]{18} - \sqrt[4]{8}.$ 9. $2 + \sqrt{2} + \sqrt{3}.$
10. $\sqrt{6} + \sqrt{3} - 1.$ 11. $\sqrt{2} + \sqrt{3} + \sqrt{5}.$ 12. $1 - \frac{1}{2}\sqrt{2} - \frac{1}{3}\sqrt{3}.$

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9. 1. $1 + \sqrt{7}.$ 2. $1 + \sqrt{5}.$ 3. $1 - \sqrt{2}.$ 4. $1 - \sqrt{3}.$
5. $1 - \sqrt{\frac{2}{3}}.$ 6. $1 - \sqrt{\frac{1}{2}}.$ 7. $\sqrt[3]{\frac{1}{4}} - \sqrt[6]{\frac{1}{2}}.$ 8. $\sqrt[3]{4} + \sqrt[6]{54}.$
9. $1 + 2\sqrt{2}.$ 10. $\dfrac{1 + \sqrt{7}}{\sqrt[3]{2}}.$ 11. $\dfrac{1 + \sqrt{2}}{\sqrt[3]{2}}.$ 12. $1 + \sqrt{3}.$

---

10. 1. $33\sqrt{-1}, \ -\frac{1}{12}\sqrt{-3}.$ 2. $4\sqrt{-1} - 10.$
3. $\sqrt{-1}, \ -\frac{3}{2}\sqrt{-1}, \ x^2 + 1.$ 4. $-\frac{8}{9}\sqrt{-2}.$
5. $x^6 + a^6.$ 6. $-12, \ -11, \ 2x + 2\sqrt{(x^2 + y^2)}.$
7. $(a^2 - b^2) + 2ab\sqrt{-1}, \ (a^3 - 3ab^2) + (3a^2b - b^3)\sqrt{-1},$
$(a^4 - 6a^2b^2 + b^4) + 4ab(a^2 - b^2)\sqrt{-1}.$
8. $6 + \sqrt{-5}, \ 3 - 4\sqrt{-1}, \ 2(1 + \sqrt{-1}).$

---

11. 1. $\pm 2, \pm 1.$ 2. $\frac{1}{4}, -\frac{1}{2}.$ 3. $\pm 2, \pm \frac{1}{2}.$ 4. $4, \frac{1}{4}.$
5. $1, \frac{1}{81}.$ 6. $\pm 5, \pm 2.$ 7. $\pm 1, \pm 3.$ 8. $\pm 8, \pm \frac{1}{8}.$
9. $-8, -\frac{1}{8}.$ 10. $\pm 1, \pm 2.$ 11. $\sqrt[n]{\frac{1}{2}}, \sqrt[n]{-\frac{1}{3}}.$
12. $\pm 1, \pm \frac{1}{2}.$ 13. $9, 116\frac{27}{81}.$ 14. $0, 3, \frac{4}{7}, -2\frac{3}{7}.$
15. $2, 3\frac{1}{2}, \frac{1}{4}(11 \pm \sqrt{-663}).$ 16. $3, \frac{1}{2}, \frac{1}{4}(7 \pm \sqrt{33}).$
17. $\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{6}(9 \pm \sqrt{-83}).$ 18 $0, a, \frac{1}{2}\{a \pm \sqrt{(5a^2 - 8ab)}\}.$
19. $3, -\frac{1}{2}, \frac{1}{4}(5 \pm \sqrt{1329}).$ 20. $2, \frac{1}{4}, \frac{1}{8}(9 \pm \sqrt{-31}).$
21. $\pm 5, \pm 4\sqrt{2}.$ 22. $-\frac{781}{1267}, -\frac{1023}{1025}.$
23. $\left(\dfrac{a + b}{a - b}\right)^{\frac{2pq}{p \sim q}}.$ 24. $(a^{-\frac{1}{2}} \pm b^{-\frac{1}{2}})^{\frac{4mn}{m-n}}.$

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12. 1. $1, 1, \frac{1}{2}(-3\pm\sqrt{5})$. 2. $1, 4, 1\frac{1}{3}, 3\frac{2}{3}$. 3. $\pm 1, \frac{1}{2}(-1\pm\sqrt{5})$.

4. $1, \frac{1}{4}\{-1\pm\sqrt{5}\pm\sqrt{(\mp 2\sqrt{5}-10)}\}$. 5. $-1, \frac{1}{2}\{1\pm\sqrt{(4p-3)}\}$.

6. $1, \frac{1}{4}(-1\pm\sqrt{-7})$. 7. $\pm\frac{1}{2}\sqrt{\left\{1-\frac{(c-2)^3}{27c}\right\}}$.

8. $-1, \frac{1}{2}\{1-p\pm\sqrt{(p^2-2p-3)}\}$.

9. $\dfrac{2na\pm\sqrt{\{(n^2+1)^2c^2-(n^2-1)^2a^2\}}}{n^2+1}$. 10. $1\frac{1}{2}, \frac{2}{3}, \frac{1}{12}(-7\pm\sqrt{-95})$.

11. $\pm 1, \frac{1}{2}(3\pm\sqrt{5})$. 12. $-1, -1, 2$. 13. $1, 4$.

14. $\frac{1}{2}(-1\pm\sqrt{5})$. 15. $\dfrac{n}{n-a}, \dfrac{n}{9n-a}$. 16. $\left(\dfrac{a^{2m}+1}{a^{2m}-1}\right)^{\pm 2}$.

17. $1, \frac{1}{2}\{2p-1\pm\sqrt{(4p^2-4p-3)}\}$.

18. $\frac{1}{3}(1\pm\sqrt{10}), \frac{1}{6}(7\pm\sqrt{85})$. 19. $-1, 1\frac{1}{3}, -\frac{1}{3}$.

20. $\pm 1, \pm\sqrt{-1}, \pm\frac{1}{2}(\sqrt{3}\pm\sqrt{-1}), \pm\frac{1}{2}(1\pm\sqrt{-3})$.

21. $2, -\frac{1}{2}, \frac{1}{2}(-1\pm\sqrt{5})$.

22. $-1, \frac{1}{2}\{p\pm\sqrt{(p^2-4)}\}$, where $p=\dfrac{-(4a+1)\pm\sqrt{\{5(4a+1)\}}}{2(a-1)}$.

23. $\pm\sqrt{\frac{1}{2}(3\pm\sqrt{41})}, \pm\sqrt{\frac{1}{2}(1\pm\sqrt{37})}$. 24. $3, -\frac{3}{4}, \frac{9}{8}(9\pm\sqrt{97})$.

25. $7\pm 4\sqrt{3}$. 26. $\dfrac{a(1+n)^2}{1+2n}, \dfrac{a(1-n)^2}{1-2n}$.

27. $\frac{1}{2}\{p\pm\sqrt{(p^2-4)}\}$, where $p=\dfrac{2a\pm\sqrt{\{2(a+1)\}}}{a-1}$.

28. $0, \pm\sqrt{(2ac-c^2)}$. 29. $a\left\{1-\dfrac{16c^2}{(c+1)^4}\right\}$. 30. $\pm\dfrac{2}{b\sqrt{(4a-b^2)}}$.

31. $\left.\begin{array}{l} x=3,\ 2,\ \frac{1}{2}(5\pm\sqrt{-151}) \\ y=2,\ 3,\ \frac{1}{2}(5\mp\sqrt{-151}) \end{array}\right\}$ 32. $\left.\begin{array}{l} x=4,\ -2,\ \ 1\pm\sqrt{-15} \\ y=2,\ -4,\ -1\pm\sqrt{-15} \end{array}\right\}$

33. $x=\frac{1}{2}[b\pm\sqrt{\{-b^2\pm\frac{2}{5}\sqrt{5(4a^5b^{-1}+b^4)}\}}]$, $y=\frac{1}{2}[b\mp\&c]$.

34. $x=1$, $y=\pm 1$; or $x=-1$ or $\frac{241}{169}$, $y=\frac{5}{13}$.

35. $x=0, y=0$; $x=\dfrac{ab}{a^2-b^2}\{-b\pm\sqrt{(2a^2-b^2)}\}$, $y=\dfrac{ab}{a^2-b^2}\{-b\mp\&c\}$.

36. $\left.\begin{array}{l} x=6,\ -4\frac{1}{2} \\ y=12,\ -9 \end{array}\right\}$ 37. $\left.\begin{array}{l} x=0,\ 4 \\ y=0,\ 3 \end{array}\right\}$ 38. $\left.\begin{array}{l} x=\pm 8 \\ y=\pm 27 \end{array}\right\}$

39. $\left.\begin{array}{l} x=8,\ -4,\ 8(19\pm 8\sqrt{6}) \\ y=4,\ \ 1,\ 8(5\pm 2\sqrt{6}) \end{array}\right\}$ 40. $\left.\begin{array}{l} x=0,\ \frac{1}{4}(1\pm\sqrt{2}) \\ y=0,\ \ \frac{1}{4} \end{array}\right\}$

41. $x=\dfrac{1}{4b}[a^2+b^2\pm\sqrt{\{10a^2b^2-3(a^4+b^4)\}}]$, $y=\dfrac{1}{4b}[a^2+b^2\mp\&c]$.

42. $\left.\begin{array}{l} x=\left(a^{m2-mn}b^{mn-n^2}c^{m}d^{-n}\right)^{\frac{1}{m^2-n^2}} \\ y=\left(a^{m2-mn}b^{mn-n^2}c^{-n}d^{m}\right)^{\frac{1}{m2-n^2}} \end{array}\right\}$

43. $\left.\begin{array}{l} x=(\sqrt{2}+1)^2,\ (\sqrt{2}-1)^2 \\ y= \quad 1, \qquad (\sqrt{2}-1)^4 \end{array}\right\}$

44. $\left.\begin{array}{l} x=5,\ 1,\ \frac{1}{29}(15\pm\ 6\sqrt{-1}) \\ y=3,\ \frac{3}{5},\ \frac{1}{29}(25\pm 10\sqrt{-1}) \end{array}\right\}$

45. $\left.\begin{array}{l} x=[b^n\{a^{\frac{1}{2}(m-n)}\pm\sqrt{(a^{m-n}-b^{m-n})}\}]^{\frac{2}{m+n}} \\ y=[a^m\{b^{\frac{1}{2}(n-m)}\mp\sqrt{(b^{n-m}-a^{n-m})}\}]^{\frac{2}{m+n}} \end{array}\right\}$

46. $\left.\begin{array}{l} x=8,\ 2 \\ y=\ 4 \\ z=2,\ 8 \end{array}\right\}$

47. $\left.\begin{array}{l} x=\sqrt[3]{(a^{-1}b^2c^2)} \\ y=\sqrt[3]{(a^2b^{-1}c^2)} \\ z=\sqrt[3]{(a^2b^2c^{-1})} \end{array}\right\}$

48. $\left.\begin{array}{l} x=\frac{3}{2},\ \frac{2}{3} \\ y=\ 1 \\ z=\frac{2}{3},\ \frac{3}{2} \end{array}\right\}$

49. $\left.\begin{array}{l} x=\quad 30,\ -35\frac{1}{2} \\ y=\pm 29,\ \pm\frac{1}{2}\sqrt{3626} \\ z=\quad 36,\ -29\frac{1}{2} \end{array}\right\}$

50. $\left.\begin{array}{l} x=9,\ 4 \\ y=\ 3 \\ z=4,\ 9 \end{array}\right\}$

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13. 1. 41, 24, 2. 2. 1, 6, 23, or 3, 3, 24. 3. 13.
4. 100, 100, 700. 5. 9. 6. 6, 1, 10.

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14. 1. $\left.\begin{array}{l} x=1,\ 8 \\ y=9,\ 2 \end{array}\right\}$ 2. $\left.\begin{array}{l} x=\ 1,\ 3,\ 4 \\ y=20,\ 3,\ 2 \end{array}\right\}$ 3. $\left.\begin{array}{l} x=11,\ 2 \\ y=\ 2,\ 11 \end{array}\right\}$ 4. $\left.\begin{array}{l} x=4 \\ y=5 \end{array}\right\}$

5. $\left.\begin{array}{l} x=\ 2,\ 4,\ 10 \\ y=16,\ 12,\ 16 \end{array}\right\}$ 6. $\left.\begin{array}{l} x=4 \\ y=3 \end{array}\right\}$ 7. $\left.\begin{array}{l} x=3 \\ y=1 \end{array}\right\}$ 8. $\left.\begin{array}{l} x=3 \\ y=4 \end{array}\right\}$

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16. 1. *min.* 7, *min.* $1\frac{3}{4}$, *max.* $\frac{1}{4}$, *max.* $\frac{1}{4}(a \sim b)^2$.
2. *max.* $\frac{1}{12}$, *min.* $2\sqrt{(ab)}$, *min.* 1, *max.* $1\frac{1}{24}$, *max.* $\frac{(a+b)^2}{4ab}$.
3. *max.* $\frac{1}{2}a^2$, *min.* $\frac{1}{2}a^2$, *min.* 2, the N° in each case being *halved.*
4. *All* N^os^ between $\frac{1}{3}$ and 3; *all* between $\frac{1}{7}$ and 7; *none* between $\frac{1}{9}$ and 1; *none* between $(\sqrt{a}\sim\sqrt{b})^2$ and $(\sqrt{a}+\sqrt{b})^2$.

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18. 1. 252, 1023. 2. 4, 56. 3. 12. 4. 3, 20.
5. 10, 92378. 6. $\frac{1}{2}m(m+1)+mn$.

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19. 1. $a^{-\frac{1}{2}}+a^{-\frac{3}{2}}x+\frac{3}{2}a^{-\frac{5}{2}}x^2+\frac{5}{2}a^{-\frac{7}{2}}x^3+\frac{35}{8}a^{-\frac{9}{2}}x^4+$ &c.

2. $a^{-\frac{2}{3}}+a^{-\frac{5}{3}}x+2a^{-\frac{8}{3}}x^2+\frac{14}{3}a^{-\frac{11}{3}}x^3+\frac{35}{3}a^{-\frac{14}{3}}x^4+$ &c.

3. $a^{\frac{3}{4}}+a^{-\frac{1}{4}}x-\frac{3}{2}a^{-\frac{5}{4}}x^2+\frac{7}{2}a^{-\frac{9}{4}}x^3-\frac{77}{8}a^{-\frac{13}{4}}x^4+$ &c.

4. $a^{\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{2}}-\frac{1}{8}a^{-\frac{1}{2}}x+\frac{1}{16}a^{-1}x^{\frac{3}{2}}-\frac{5}{128}a^{-\frac{3}{2}}x^2+$ &c.

5. $1+x+2x^2+3x^3+5x^4+$ &c. 6. $1+2x+5x^2+10x^3+20x^4+$ &c.

7. $1+\frac{1}{2}x-\frac{5}{8}x^2+\frac{5}{16}x^3-\frac{45}{128}x^4+$ &c. 8. $1-x-\frac{2}{3}x^2-x^3-\frac{16}{9}x^4-$ &c.

9. $a^{-\frac{4}{3}}+\frac{2}{3}a^{-\frac{7}{3}}x+\frac{11}{9}a^{-\frac{10}{3}}x^2+\frac{130}{81}a^{-\frac{13}{3}}x^3+\frac{605}{243}a^{-\frac{16}{3}}x^4+$ &c.

10. $a^{-6}-6a^{-7}x+27a^{-8}x^2-104a^{-9}x^3+306a^{-10}x^4-$ &c.

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20. 1. $+\dfrac{8.7\ldots(9-r)}{1.2\ldots r}\,x^r$. 2. $+\dfrac{12.11\ldots(13-r)}{1.2\ldots r}\,x^r$.

3. $(-1)^r\,.\,\dfrac{9.8\ldots(10-r)}{1.2\ldots r}\,a^{9-r}x^r$. 4. $+\dfrac{7.6\ldots(8-r)}{1.2\ldots r}\,(3x)^{7-r}y^r$.

5. $(-1)^r.x^r$. 6. $(-1)^r.(r+1)\,3^rx^r$. 7. $+\dfrac{(r+1)\,(r+2)}{1.2}\,2^rx^r$.

8. $+\dfrac{3.5.7\ldots(r+1)}{1.2.3\ldots r.2^r}\,x^r$. 9. $+\dfrac{1.4.7\ldots(3r-2)}{1.2.3\ldots r.3^r}\,x^r$.

10. $(-1)^r\,.\,\dfrac{1.3.5\ldots(2r-1)}{1.2.3\ldots r.2^r}\,x^{2r}$. 11. $+\dfrac{r+1}{2^{r+2}}\,x^r$.

12. $(-1)^r\,.\,\dfrac{b^r}{a^{r+1}}\,x^r$. 13. $(-1)^r\,.\,\dfrac{(r+1)\,(r+2)}{1.2}\,a^{1+\frac{r}{3}}b^{-\frac{r}{3}}$.

14. $(-1)^{r+1}\,.\,\dfrac{1.1.3\ldots(2r-3)}{1.2.3\ldots r}\,x^r$. 15. $-\dfrac{2.1.4\ldots(3r-5)}{1.2.3\ldots r}\,x^r$.

16. $+a^{\frac{2}{3}}\,.\,\dfrac{4.1.2.5\ldots(3r-7)}{1.2.3.4\ldots r.3^r}\,.\,\dfrac{x^{2r}}{a^{2r}}$.

17. $+a^{-\frac{3}{2}}\,.\,\dfrac{3.7.11\ldots(4r-1)}{1.2.3\ldots r.4^r}\,.\,\dfrac{x^{2r}}{a^{2r}}$.

18. $+(ax)^{-\frac{1}{3}}\,.\,\dfrac{1.4.7\ldots(3r-2)}{1.2.3\ldots r.3^r}\,.\,\dfrac{x^r}{a^r}$.

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21. 1. $3^{\text{rd}}=3\frac{3}{4}$. 2. $5^{\text{th}}=6^{\text{th}}=7\frac{7}{8}$. 3. $1^{\text{st}}=2^{\text{nd}}=1$.
4. $2^{\text{nd}}=2$. 5. $5^{\text{th}}=4\frac{67}{125}$. 6. $4^{\text{th}}=5^{\text{th}}=17\frac{341}{507}$.

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22. 1. $1\frac{1}{6}$. 2. $\frac{1}{4}\sqrt{6}$. 3. na^{n-1}. 4. $\frac{1}{8}$. 5. $1\frac{1}{2}$. 6. $\frac{1}{3}$.

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23. 1. $a^5+5a^4bx+(10a^3b^2+5a^4c)\,x^2+(10a^2b^3+20a^3bc)\,x^3$
$+(5ab^4+30a^2b^2c+10a^3c^2)\,x^4$.

2. $a^7-\frac{7}{2}a^6bx+(\frac{21}{4}a^5b^2+\frac{7}{3}a^6c)\,x^2-(\frac{35}{8}a^4b^3+7a^5bc)\,x^3$
$+(\frac{35}{16}a^3b^4+\frac{35}{4}a^4b^2c+\frac{7}{3}a^5c^2)\,x^4$.

3. $1+8x+12x^2-40x^3-70x^4$. 4. $1-4x+6x^2-4x^3+x^4$.

5. $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\frac{5}{128}x^4$. 6. $x^{-3}+\frac{3}{2}x^{-2}+\frac{1}{2}x^{-1}+\frac{1}{240}x$.

7. $1+3x+9x^2+27x^3+81x^4$.

8. $a^{-\frac{2}{3}}+\frac{2}{3}a^{-\frac{5}{3}}bx+(\frac{5}{9}a^{-\frac{8}{3}}b^2+\frac{2}{3}a^{-\frac{5}{3}}c)\,x^2+(\frac{40}{81}a^{-\frac{11}{3}}b^3+\frac{10}{9}a^{-\frac{8}{3}}bc)\,x^3$
$+(\frac{110}{243}a^{-\frac{14}{3}}b^4+\frac{40}{27}a^{-\frac{11}{3}}b^2c+\frac{5}{9}a^{-\frac{8}{3}}c^2)\,x^4$. 9. $1+3x+3x^2+x^3$.

10. $(30a^4bc-20a^3b^3)\,x^3+(15a^4c^2-60a^3b^2c+15a^2b^4)\,x^4$.

11. $(\frac{1}{2}a^{-\frac{3}{2}}c+\frac{3}{8}a^{[illegible]}b^2)\,x^2+(\frac{1}{2}a^{-\frac{3}{2}}d-\frac{3}{4}a^{-\frac{5}{2}}bc-\frac{5}{16}a^{-\frac{7}{2}}b^3)\,x^3$.

12. $\frac{44}{945}$, $-\frac{4208}{14175}$.

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24. 1. .9030900, .9542426, 1.0791813, 1.3010300, 1.3979400, 1.7781513.

2. $\bar{1}.5228787$, $\bar{1}.3979400$, $\bar{1}.6020600$, $\bar{2}.4771213$, $\bar{2}.5228787$, $\bar{3}.5185140$.

3. .2552726, 2.1461280, .1583626, $\bar{2}.7958800$, $\bar{4}.9172146$.

4. .0211893, 1.0253059, .6232493, .6434527, $\bar{2}.0453230$.

5. .0880456, .0416462, $\bar{1}.5000000$, $\bar{1}.4336766$, .0527875.

6. $\bar{1}.8465449$. 7. .8923879.

8. $\bar{3}.4647060$, $\bar{1}.8957576$, $\bar{1}.5355396$, $\bar{1}.4982833$.

9. $\bar{1}.8035760$. 10. $\bar{1}.8149140$.

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25. 1. 1.5527807, $\bar{1}.5530013$, $\bar{3}.5528198$, .5530092.

2. 3571.693, .3573465, .00035709, .3572892. 3. 1.3010502, .0200087.

4. 6.4913649, 4.4913652, 1.4913722.

5. 4.498527. 6. .5738614.

Table for Ex. 3.

D.	Pro.	
217		217
	1	22
	2	43
	3	65
	4	87
	5	109
	6	130
	7	152
	8	174
	9	195

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26. 1. $\dfrac{\log c - \log a}{\log b}$. 2. $\dfrac{\log(\log c) - \log(\log a)}{\log b}$.

3. 4.153. 4. $\dfrac{ma \log b}{n(m \log b - \log a)}$.

5. $x = \dfrac{m \log c}{m \log a + n \log b}$, $y = \dfrac{n \log c}{m \log a + n \log b}$. 6. -1.

7. 2 or -1. 8. $1\frac{1}{3}$. 9. $x = (ab^{-1})^{\frac{b}{a-b}}$, $y = (ab^{-1})^{\frac{a}{a-b}}$.

10. $\dfrac{\log(1 \pm \sqrt{5}) - \log 2}{2 \log a}$. 11. $\dfrac{\log\{b \pm \sqrt{(b^2-4)}\} - \log 2}{\log a}$.

12. $x = (-\frac{1}{2} \pm \frac{1}{6}\sqrt{57})^5$; $y = (-\frac{1}{2} \pm \frac{1}{6}\sqrt{57})^4$.

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27. 1. $.213 \times 2.03 \times 3.\dot{1} = .44043 \times 3\frac{1}{5} = 2.3$. 2. $\frac{34}{400}$, $\frac{491}{1176}$.

3. $113.\dot{1}10132232\dot{0}$, 27.243656 &c, 1*e*.3*toe*62 &c.

4. 0000001010001 &c, .0000210212 &c, $.\dot{0}123456789\dot{t}$.

5. $.332 \times 2.31043 \times .41 = 1.3 \times .41 = 1.013$ *sen.* $= 1.06$ *duod.*; $.714 \times 2.646 \times .84 = 1.6 \times .84 = 1.06$ *duod.* $= 1.013$ *sen.*

6. $\frac{645}{6600}$, $\frac{646}{7700}$, $\frac{647}{8800}$ $\frac{648}{9900}$.

7. $277\frac{7}{9}$ sq. ft., $16\frac{2}{3}$ ft, 23.57 ft. nearly. 8. $52\frac{1}{12}$.

9. $13\frac{1}{12}$ ft. 10. 51. 11. $40\frac{25}{32}$ cub. ft. 12. 28.

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28. 1. 48 yrs nearly. 2. £81 18*s* 7*d*. 3. 25 yrs; 18 yrs nearly.
4. £119 19*s* 6*d* nearly. 5. £3121 12*s*.
6. £2775 1*s* 10*d* nearly. 7. 25 yrs.
8. £20,000; £17,276 15*s* nearly. 9. £1,600.
10. £27,657 nearly. 11. $\dfrac{P(R^{m+n}-1)}{R^m(R^n-1)}$.
12. £1,440, £921 12*s*.

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29. 1. $1+\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4}$. 2. $2+\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5}$. 3. $\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5+}\,\frac{1}{6}$.
4. $3+\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3}$. 5. $\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{4}$.
6. $3+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3}$. 7. $\frac{1}{5+}\,\frac{1}{2+}\,\frac{1}{7+}\,\frac{1}{3}$.
8. $2+\frac{1}{4+}\,\frac{1}{6+}\,\frac{1}{8}$. 9. $\frac{1}{2+}\,\frac{1}{5+}\,\frac{1}{7+}\,\frac{1}{5+}\,\frac{1}{2}$.
10. $1+\frac{1}{4+}\,\frac{1}{5+}\,\frac{1}{7+}\,\frac{1}{9}$. 11. $\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{6+}\,\frac{1}{8+}\,\frac{1}{10}$.
12. $5+\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{7}$.

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30. 1. $2+\frac{1}{4+}\,\frac{1}{4+\&c.}$ 2. $2+\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{4+\&c.}$
3. $2+\frac{1}{1+}\,\frac{1}{4+\&c.}$ 4. $3+\frac{1}{3+}\,\frac{1}{6+\&c.}$
5. $3+\frac{1}{2+}\,\frac{1}{6+\&c.}$ 6. $3+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+\&c.}$
7. $4+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{8+\&c.}$
8. $4+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{8+\&c.}$
9. $4+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{8+\&c.}$
10. $4+\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{8+\&c.}$ 11. $5+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{10+\&c.}$
12. $7+\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{14+\&c.}$

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31. 1. $\frac{1}{4}, \frac{7}{29}, \frac{64}{265}$. 2. $2, \frac{9}{4}, \frac{92}{41}, \frac{285}{127}$. 3. $\frac{1}{2}, \frac{3}{7}, \frac{13}{30}, \frac{68}{157}, \frac{421}{972}$.
4. $1, \frac{5}{4}, \frac{6}{5}, \frac{11}{9}, \frac{17}{14}, \frac{45}{37}, \frac{152}{125}, \frac{197}{162}, \frac{743}{611}$. 5. $2, \frac{7}{3}, \frac{37}{16}, \frac{266}{115}, \frac{2431}{1051}$.
6. $\frac{1}{8}, \frac{3}{25}, \frac{25}{208}, \frac{153}{1273}, \frac{1096}{9119}$. 7. $\frac{117}{37}$. 8. $\frac{333}{89}$.
9. $\frac{244}{63}$. 10. $\frac{268}{65}$. 11. $\frac{140}{33}$. 12. $\frac{161}{36}$.

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32. 1. Two; four; four. 5. 1.3130. 6. .000085; .00012.
7. 23. 9. ± 1.52; 1.43, .23.
10. $p - \frac{1}{2}r + \frac{1}{2}\sqrt{(r^2 + 4q^{-1}r)}$; $p + \dfrac{1}{q - \frac{1}{2}s + \frac{1}{2}\sqrt{(s^2 + 4r^{-1}s)}}$.
11. 2.303; $-\frac{4}{5}$; $\frac{19}{17}$. 12. $x^2 - ax - b = 0$.

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33. 1. $\left.\begin{matrix} x = 2,\ 7,\ 26,\ \&c \\ y = 1,\ 4,\ 15,\ \&c \end{matrix}\right\}$ 2. $\left.\begin{matrix} x = 3,\ 17,\ 99,\ 577,\ \&c \\ y = 1,\ 6,\ 35,\ 204,\ \&c \end{matrix}\right\}$
3. $\left.\begin{matrix} x = 5,\ 235,\ \&c \\ y = 1,\ 49,\ \&c \end{matrix}\right\}$ 4. $\left.\begin{matrix} x = 4,\ 19,\ 211,\ \&c \\ y = 1,\ 4,\ 44,\ \&c \end{matrix}\right\}$
5. $\left.\begin{matrix} x = 18,\ \&c \\ y = 5,\ \&c \end{matrix}\right\}$ 6. $\left.\begin{matrix} x = 6,\ 126,\ \&c \\ y = 1,\ 19,\ \&c \end{matrix}\right\}$ 7. $\left.\begin{matrix} x = 20,\ 398,\ \&c \\ y = 3,\ 60,\ \&c \end{matrix}\right\}$
8. $\left.\begin{matrix} x = 453,\ \&c \\ y = 58,\ \&c \end{matrix}\right\}$ $\left.\begin{matrix} x = 164,\ \&c \\ y = 21,\ \&c \end{matrix}\right\}$

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34. 1. $\frac{1}{2} + \frac{3}{4}x + \frac{9}{8}x^2 + \frac{27}{16}x^3 + \frac{81}{32}x^4 + \&c.$
2. $\frac{1}{2} - \frac{1}{4}x + \frac{3}{8}x^2 - \frac{9}{16}x^3 + \frac{27}{32}x^4 - \&c.$
3. $1 - x^2 - x^3 + x^5 + x^6 - \&c.$ 4. $1 + 4x + 2x^2 - 8x^3 - 22x^4 - \&c.$
5. $y - y^2 + y^3 - y^4 + y^5 - \&c.$ 6. $y + 2y^2 + 5y^3 + 14y^4 + 42y^5 + \&c.$
7. $y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \frac{1}{24}y^4 + \&c.$ 8. $y + \frac{1}{3}y^3 + \frac{2}{15}y^5 + \frac{17}{315}y^7 + \&c.$
9. $y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \&c.$ 10. $y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7 + \&c.$
11. $y - \frac{1}{6}y^3 + \frac{1}{24}y^5 - \frac{61}{5040}y^7 + \&c.$ 12. $y + \frac{1}{6}y^3 + \frac{1}{24}y^5 + \frac{61}{5040}y^7 + \&c.$

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35. 1. $\dfrac{2}{x+3} - \dfrac{1}{x+2}$. 2. $\dfrac{3}{2(x-2)} - \dfrac{1}{2x}$. 3. $\dfrac{5}{2(x-1)} - \dfrac{1}{2(x+1)}$.
4. $-\dfrac{3}{x} + \dfrac{4}{(x+1)^2} + \dfrac{3}{x+1}$. 5. $\dfrac{1}{(x-1)^2} + \dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$.
6. $\dfrac{1}{x^2} - \dfrac{2}{x} + \dfrac{3}{x+1}$. 7. $\dfrac{1}{x^3} - \dfrac{1}{x^2} + \dfrac{3}{x} - \dfrac{3}{x+1}$.
8. $\dfrac{5}{x} - \dfrac{1}{x^2} - \dfrac{5}{x+1} - \dfrac{4}{(x+1)^2}$. 9. $\dfrac{1}{x^2} + \dfrac{1}{x^3} - \dfrac{1}{(x-1)^2}$.
10. $\dfrac{1}{4}\left\{\dfrac{1}{(x+1)^3} - \dfrac{1}{(x+1)^2}\right\} + \dfrac{1}{16}\left\{\dfrac{1}{x-1} - \dfrac{1}{x+1}\right\} + \dfrac{1}{8(x-1)^2}$.

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36. 1. $\dfrac{3x+2}{x^2+1}-\dfrac{3}{x}.$ 2. $\dfrac{3}{4(x+1)}-\dfrac{1}{4(x-1)}-\dfrac{x-2}{2(x^2+1)}.$

3. $\dfrac{2}{3(x-1)}+\dfrac{x-1}{3(x^2+x+1)}.$ 4. $\dfrac{1}{3(x+1)}+\dfrac{5x-4}{3(x^2-x+1)}.$

5. $\dfrac{1}{3(x-1)}-\dfrac{1}{x}+\dfrac{2x+1}{3(x^2+x+1)}.$

6. $-\dfrac{2}{x^2}+\dfrac{3}{x}-\dfrac{3x-2}{(x^2+1)^2}-\dfrac{3x-2}{x^2+1}.$

7. $\dfrac{3}{x}-\dfrac{x}{(x^2-x+1)^2}-\dfrac{3x-6}{x^2-x+1}.$

8. $\dfrac{1}{16}\left\{\dfrac{1}{(x-1)^2}+\dfrac{1}{x-1}+\dfrac{1}{(x+1)^2}-\dfrac{1}{x+1}\right\}+\dfrac{1}{4}\left\{\dfrac{1}{(x^2+1)^2}-\dfrac{1}{x^2+1}\right\}$

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37. 1. $\dfrac{1+3x}{1-x-x^2}.$ 2. $\dfrac{1+4x}{1-2x+3x^2}.$ 3. $\dfrac{1+2x-x^2}{1-2x+x^2-3x^3}.$

4. $\dfrac{1+x+2x^2}{1-2x+x^2-2x^3}.$ 5. $\dfrac{1-x}{(1+x)^2}$, $(-1)^n.(2n+1)\,x^n.$

6. $\dfrac{1+2x+3x^2}{(1-x)^3}$, $(3n^2+n+1)\,x^n.$

7. $\dfrac{8\{1-(3x)^n\}}{5(1-3x)}-\dfrac{3\{1-(-2x)^n\}}{5(1+2x)}.$

8 $\dfrac{7\{1-(-2x)^n\}}{3(1+2x)}-\dfrac{1-x^n}{3(1-x)}.$ 9. $2^{n+1}-n-2.$

10. $\frac{1}{4}(3^n-1)+\frac{1}{2}n.$ 11. $\frac{1}{6}\{2^{n+3}-9+(-1)^n\}.$ 12. $\frac{5}{2}n-\frac{9}{4}(1-3^{-n}).$

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38. 1. $\dfrac{1-x^n}{(1-x)^2}-\dfrac{nx^n}{1-x}$, $\dfrac{1}{(1-x)^2}.$

2. $\dfrac{1-(3n-2)\,x^n}{1-x}+\dfrac{3x(1-x^{n-1})}{(1-x)^2}.$ 3. $(2n-3)\,2^n+3.$

4. $\frac{1}{9}\{2+(6n+1)(-\frac{1}{2})^{n-1}\}$, $\frac{2}{9}.$ 5. $(2^n-1)\,a+\{(n-2)\,2^n+2\}\,x.$

6. $\dfrac{a\{1-(-x)^n\}}{1+x}-\dfrac{x+\{n+(n-1)\,x\}\,(-x)^n}{(1+x)^2}$, $\dfrac{a}{1+x}-\dfrac{x}{(1+x)^2}$

7. $\frac{1}{3}n\,(n+1)\,(n+2).$ 8. $\frac{1}{6}n\,(n+1)\,(2n+7).$

9. $\dfrac{n}{n+1}$, 1. 10. $\dfrac{n\,(n+3)}{4\,(n+1)\,(n+2)}$, $\dfrac{1}{4}.$

11. $\dfrac{n\,(3n+5)}{4\,(n+1)\,(n+2)}$, $\dfrac{3}{4}.$ 12. $\dfrac{n\,(5n+7)}{4\,(n+1)\,(n+2)}$, $1\frac{1}{4}.$

13. $\frac{1}{3}n\,(4n^2-1).$ 14. $\frac{1}{30}n\,(n+1)\,(2n+1)\,(3n^2+3n-1).$

15. $n^2\,(2n^2-1).$ 16. $\frac{1}{12}n\,(n+1)\,(n+2)\,(3n+1).$

17. $\frac{n}{4(n+1)}, \frac{1}{4}$. 18. $\frac{3}{80} - \frac{2n+9}{12(n+4)(n+5)}, \frac{3}{80}$.

19. $\frac{5}{36} - \frac{3n+5}{6(n+1)(n+2)(n+3)}, \frac{5}{36}$. 20. $\frac{n(n+1)}{n+2}$.

21. 2925, 5525. 22. 2705, 5140. 23. 12700. 24. 11270.

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41. 1. 12, 12, 15, 20, 18, 16. 2. 8, 6, 12, 16, 12, 12; 4, 0, 9, 12, 6, 8.

3. 6, 6, 8; 4, 4, 2. 4. 168, 234, 403, 144, 156, 372, 96, 0, 279.

5. 16, 24, 48, 80, 400.

6. 8, 24, 24, 32, 40, 200; $60p \pm 1$, $60p \pm 7$, $60p \pm 11$, $60p \pm 13$, $60p \pm 17$, $60p \pm 19$, $60p \pm 23$, $60p \pm 29$.

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42. 1. $x = \frac{m^2 - 29n^2}{4mn} = \frac{5}{7}$, if $m = 7$, $n = 1$.

2. $x = \frac{2mn - 5n^2}{m^2 - 7n^2} = 3$, if $m = 8$, $n = 3$.

3. $x = \frac{m^2 + n^2}{3m^2 - 2n^2} = 2$, if $m = 1$, $n = 1$.

4. $x = \frac{m^2 + n^2}{m^2 - 2mn - n^2} = 5$, if $m = 3$, $n = 1$.

5. $x = \frac{m^2 + 2n^2}{m^2 - 2n^2} = 3$, if $m = 2$, $n = 1$.

6. $x = \frac{m^2 + 4mn + 2n^2}{m^2 - 2n^2} = 41$, if $m = 3$, $n = 2$.

7. $x = \frac{7n^2 - 2mn}{m^2 - 10n^2} = 24$, if $m = 19$, $n = 6$.

8. $x = \frac{m^2 - 8mn + 5n^2}{m^2 - 5n^2} = 7$, if $m = 2$, $n = 1$.

9. $x = \frac{(m+n)^2}{m^2 - 2n^2} = 25$, if $m = 3$, $n = 2$.

10. $x = \frac{2m^2 - 10mn + 13n^2}{m^2 - 6n^2} = 2$, if $m = 5$, $n = 2$.

11. $x = \frac{m^2 - 2mn + 2n^2}{m^2 - 2n^2} = 5$, if $m = 3$, $n = 2$.

12. $x = \frac{m^2 + 6mn + 7n^2}{m^2 - 7n^2} = 271$, if $m = 8$, $n = 3$.

13. $x = \frac{2m^2 - 4mn - n^2}{m^2 - 6n^2} = 6$, if $m = 5$, $n = 2$.

14. $x = \frac{m^2 - 6mn + 2n^2}{m^2 - 5n^2} = 6$, if $m = 2$, $n = 1$.

15. $x = \dfrac{3(m+2n)^2}{m^2-6n^2} = 243$, if $m=5$, $n=2$.

16. $x = \dfrac{m^2-4mn+7n^2}{m^2-7n^2}$ or $\dfrac{2m^2-10mn+14n^2}{m^2-7n^2}$

$= 31$ or 14, if $m=8$, $n=3$.

17. $x=2mn+7n^2$, $y=m^2-11n^2$; if $m=4$, $n=1$, then $x=15$, $y=5$.

18. $x=m^2-6n^2$, $y=4mn+5n^2$; if $m=3$, $n=1$, then $x=3$, $y=17$.

19. $x=m^2+3n^2$, $y=3m^2-2n^2$; if $m=1$, $n=1$, then $x=4$, $y=1$.

20. $x=m^2+4mn+2n^2$, $y=m^2-5n^2$; if $m=3$, $n=1$, then $x=23$, $y=4$.

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43. 1. $z = \dfrac{24m^2+n^2}{2mn}$, $x=z^2-11$; if $m=1$, $n=2$ or 3, then $x=14$; if $m=2$, $n=1$, then $x=38$.

2. $x = \dfrac{(m^2+n^2)^2}{4mn(m^2-n^2)} = \dfrac{25}{24}$, if $m=2$, $n=1$.

3. $x=\frac{1}{5}(2p^2+q^2)$, $y=\frac{1}{5}(p^2-2q^2)$; or, (putting $p=5m$, $q=5n$,) $x=5(2m^2+n^2)$, $y=5(2m^2-n^2)$; if $m=2$, $n=1$, then $x=45$, $y=10$. 4. $x=\frac{1}{28}$. 5. $x=\frac{1}{11}(m^2-3)=2$, if $m=5$.

6. $x=3$. 7. $x=3\frac{1}{4}$. 8. $x=2$, $\frac{1}{3}$, $\frac{6}{7}$, $-\frac{8}{25}$, $\frac{56}{103}$, $\frac{2675}{4448}$.

9. $x=2\frac{17}{36}$. 10. $x=-3$. 11. $x=-\frac{2}{5}$. 12. $x=1\frac{2}{11}$.

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44. 1. $x = \dfrac{m^4+4n^4}{4m^2n^2} = \dfrac{5}{4}$, if $m=n=1$.

2. $x = \dfrac{n^2}{2m^2-n^2} = 8$, if $m=3$, $n=4$.

3. $x = \dfrac{m^2+n^2a}{2mn}$, $y = \dfrac{m^2-n^2a}{2mn}$: when $a=15$, put $m=5$, $n=1$ then $x=4$, $y=1$: when $a=16$, put $m=8$, $n=1$; then $x=5$, $y=3$.

4. $x=\frac{1}{2}(m^2+n^2)$, $y=\frac{1}{2}(m^2-n^2)$, where m and n must be taken both even or both odd: if $m=3$, $n=1$, then $x=5$, $y=4$.

5. $x=(m^2-n^2)^2$, $y=4m^2n^2$; if $m=2$, $n=1$, then $x=9$, $y=16$.

6. $x = \dfrac{m^2+n^2}{2mn}a$, $y = \dfrac{m^2-n^2}{2mn}a$; if $a=21$, $m=3$, $n=1$, then $x=35$, $y=28$.

7. $x=m^2+n^2$, $y=2mn$; if $m=2$, $n=1$, then $x=5$, $y=4$.

8. $x=m^2-n^2$, $y=2mn+n^2$; if $m=2$, $n=1$, then $x=3$, $y=5$.

9. $x = \dfrac{m^2-n^2}{m^2+n^2}a$, $y = \dfrac{2mn}{m^2+n^2}a$; if $a=15$, $m=2$, $n=1$, then $x=9$, $y=12$. 10. $x=m^2+n^2a$, $y=2mn$, $z=m^2-n^2a$.

11. $x = \dfrac{m^2+1}{m^2n^2-1}$, $y = \dfrac{n^2+1}{m^2n^2-1}$; if $m=2$, $n=1$, then $x=\frac{5}{3}$, $y=\frac{2}{3}$.

12. $x = \dfrac{(m^2-n^2)\,a + 2mnb}{m^2+n^2}$, $y = \dfrac{(m^2-n^2)\,b - 2mna}{m^2+n^2}$.

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45. 1. $\frac{2}{7}$, $\frac{5}{8}$, $\frac{3}{7}$, or $\frac{16}{56}$, $\frac{35}{56}$, $\frac{24}{56}$. 2. 2 : 3. 3. $\frac{4}{13}$, $\frac{2}{13}$.
4. 34 : 43. 5. $\frac{3}{8}$, $\frac{4}{9}$, $\frac{3}{28}$, $\frac{5}{14}$, $\frac{15}{28}$.
6. $\frac{1}{4}$, $\frac{1}{3}$, $\frac{5}{12}$, $\frac{2}{11}$, $\frac{5}{33}$, $\frac{19}{33}$; $\frac{3}{11}$, $\frac{7}{22}$, $\frac{27}{55}$.
7. $\frac{2}{15}$, $\frac{3}{10}$, $\frac{5}{21}$, $\frac{27}{42}$. 8. $\frac{2}{11}$, $\frac{1}{110}$.

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46. 1. 3 : 5, 1 : 7. 2. 6 : 1. 3. 3 : 29, 15 : 49.
4. 2 or 3 times, chance for each $\frac{5}{16}$.
5. $\frac{8}{125}$, $\frac{12}{125}$, $\frac{36}{125}$; $\frac{1}{30}$, $\frac{1}{10}$, $\frac{3}{10}$. 6. $\frac{96}{625}$, $\frac{112}{625}$, $\frac{608}{625}$, $\frac{256}{625}$.
7. 3 or 4 black, chance for each $\frac{256}{625}$;
2 or 3 black, chance for each $\frac{216}{625}$.
8. $\frac{5}{8}$, $\frac{35}{64}$. 9. 3 red, 2 white. 10. $\frac{125}{3888}$, $\frac{125}{3888}$, $\frac{3875}{3888}$.
11. $\frac{14}{75}$, $\frac{1}{25}$, $\frac{1}{27}$, $\frac{7}{45}$. 12. $\frac{28}{1125}$, $\frac{98}{1215}$, $\frac{1}{81}$, $\frac{7}{243}$.

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47. 1. $3\frac{1}{2}d$, $7d$. 2. $15\frac{17}{18}d$, $4\frac{5}{18}d$. 3. $1\frac{2}{7}d$, $7\frac{5}{7}d$.
4. $2\frac{4}{7}d$, $15\frac{3}{7}d$, in either case. 5. $6\frac{3}{4}d$, $5\frac{1}{4}d$.
6. $4s\ 2d$, $8s\ 4d$, $12s\ 6d$. 7. $30\frac{5}{8}s$, $56\frac{7}{8}s$, $17\frac{1}{2}s$.
8. $52\frac{1}{2}s$ each; A and B, $39\frac{3}{8}s$ each, C, $26\frac{1}{4}s$; $26\frac{1}{4}s$ each.
9. $6s\ 8d$, $3s\ 4d$; $6s\ 6d$, $3s\ 6d$.
10. $+\,2\frac{32}{49}s$, $-\,\frac{20}{49}s$, $-\,2\frac{12}{49}s$; $+\,2s\ 6d$, $-\,\frac{3}{7}s$, $-\,2\frac{3}{14}s$.
11. $1\frac{1}{5}d$ in either case. 12. A, $+2\frac{1}{7}s$, B, $-\frac{3}{14}s$, C, $-1\frac{13}{14}s$.

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48. 1. $\frac{4}{5}$, $\frac{17}{20}$. 2. $\frac{3}{20}$, $\frac{1}{10}$; $\frac{1}{20}$. 3. $\frac{4}{63}$. 4. $\frac{1}{2}$, $\frac{27}{50}$.
5. $\frac{1}{20}$, $\frac{27}{50}$; $\frac{5}{22}$, $\frac{28}{55}$; $\frac{5}{22}$, $\frac{28}{55}$. 6. $\frac{4}{5}$, $\frac{17}{20}$.
7. $\frac{2}{7}$, $\frac{11}{14}$. 8. $\dfrac{2n-1}{3n}$, $\dfrac{2n+1}{3n}$. 9. $\frac{3}{4}$, £1 10s 6d.
10. £1 14s 8d, £1 8s 4d; £1 19s $8\frac{4}{5}d$, £1 3s $3\frac{1}{5}d$;
£1 11s 6d each.

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49. 1. $\frac{1}{9}$, $\frac{1}{6}$, $\frac{1}{9}$, $\frac{1}{12}$; $\frac{1}{36}$, $\frac{5}{72}$, $\frac{25}{216}$, $\frac{1}{8}$.
2. $\frac{5}{108}$, $\frac{7}{72}$, $\frac{1}{8}$, $\frac{1}{36}$; $\frac{10}{1296}$, $\frac{35}{1296}$, $\frac{104}{1296}$, $\frac{125}{1296}$. 3. $\frac{1}{2}$ in each case.
4. 7 : 65. 5. $\frac{6}{7}$, $\frac{2}{5}$, $\frac{7}{12}$. 6. $\frac{27}{35}$, $\frac{4}{5}$. 7. $\frac{63}{125}$, $\frac{63}{105}$.
8. $\frac{7}{15}$. 9. £$7\frac{1}{2}$, £$5\frac{5}{7}$, £$5\frac{1}{3}$. 10. $\frac{169}{313}$, $\frac{42}{74}$, $\frac{63}{113}$.
11. $\frac{4}{5}$, $\frac{1}{10}$. 12. $\frac{29}{60}$, $\frac{3}{7}$. 13. $\frac{3}{8}$, $\frac{9}{14}$, $\frac{8}{53}$.
14. $\frac{16}{17}$, $\frac{41}{81}$, $\frac{196}{365}$, $\frac{1}{2}$. 15. $\frac{16}{17}$, $\frac{17}{25}$. 16. $\frac{1}{2}$, $\frac{4}{13}$, $\frac{31}{60}$, $\frac{63}{113}$.

MISCELLANEOUS EXAMPLES: PART II.

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1. $2x^4 + (2a - 3b + 2c)\,x^3 - (3ab - 2ac + 3bc + 2b)\,x^2$
   $- (3abc - 3b^2 + 2bc)\,x + 3b^2c.$

2. $a^{\frac{2}{3}}x^{-\frac{2}{3}} + a^{-\frac{2}{3}}x^{\frac{2}{3}} + a^{\frac{1}{3}}x^{-\frac{1}{3}} + a^{-\frac{1}{3}}x^{\frac{1}{3}}.$ 4. 4.

5. $x = 5,\ y = 3,\ z = 1;$ or $x = 4,\ y = 1,\ z = 2.$

6. $p^2 - q,\ p\,(3q - p^2),\ (p^2 - q)\,(p^2 - 3q).$

7. $\dfrac{2a}{n\,(n+1)},\ \dfrac{4a}{n\,(n+1)},\ \dfrac{6a}{n\,(n+1)}$, &c. 8. 15.

9. 0 or $\dfrac{2a^3}{(a^2+1)\,c}$. 10. A's share is $\dfrac{PR^{b+c}}{R^{a+b} + R^{a+c} + R^{b+c}}$.

11. $\dfrac{y^2 - (a+b)\,y + ab}{3y - (a+2b)}$. 12. $\frac{1}{2}\left(xy + \dfrac{1}{xy}\right)$. 13. $2\,(n - \frac{1}{3})$.

14. $5s$ and $4d$ per lb. 16. 180. 17. 2 gals, 12 gals.

18. $\sqrt[4]{\dfrac{1+c}{4\,(1-c)}} + \sqrt[4]{\dfrac{1-c}{4\,(1+c)}}$.

19. $\frac{1}{2}\{p^2 \pm p\sqrt{(p^2+4a)}\}$, where $p = \frac{1}{2}\{-a \pm \sqrt{(a^2+4b)}\}$. 20. $\sqrt{\{a(a+c)\}}$.

21. $\dfrac{(a+b+c-d)\ (b+c+d-a)\ (c+d+a-b)\ (d+a+b-c)}{4\,(ab+cd)^2}$.

22. $nx^2 - 2mx + m = 0.$ 24. 1 : 3. 25. 15. 26. 22.

27. 4, 59, 55. 28. 7 years. 29. $x = \pm\, 2\frac{1}{2},\ y = \pm\, 1\frac{1}{2}$.

30. $\dfrac{\log 2}{\log\,(mn - m + n) - \log mn}$.

31. (i) $\pm\sqrt{\{\frac{1}{2}(a+b)\}} \pm \sqrt{\{\frac{1}{2}(a-b)\}}$; (ii) $\pm\frac{1}{2}a\,\{\sqrt{(1+n+n^2)\}} \pm \sqrt{(1-n+n^2)}\}$.

32. (i) $pr^3 = q^3s,\ q^2 = 3pr$; (ii) $p^2s = r^2,\ pq = \frac{1}{4}p^3 \pm 2r$.

33. 100 miles. 35. 1506.

36. $\dfrac{1}{n}\left\{a^n - na^{n-2} + \dfrac{n\,(n-3)}{1.2}\,a^{n-4} - \dfrac{n\,(n-4)\,(n-5)}{1.2.3}\,a^{n-6} + \text{\&c.}\right\}$.

37. 4. 38. .3979400, .3521825, $\bar{1}$.3467875, $\bar{1}$.8494850, $\bar{1}$.8409595, .3737275. 39. $x = (\sqrt{a} \pm \sqrt{b})^2,\ y = (\sqrt{a} \mp \sqrt{b})^2$.

40. $\frac{7}{2592},\ \frac{469}{5184}$. 42. -1. 43. $\frac{1}{2}\,(5n - 4p),\ \frac{1}{2}\,(4p - 3n)$.

44. $a^{-\frac{1}{xy}} + a^{\frac{1}{2yz}}b^{\frac{1}{2z}}$. 45. 59 and 16, or 29 and 46.

46. $-\dfrac{2.1.4.7\ldots(3r-5)}{1.2.3.4\ldots\ r.3^r}\,a^{\frac{4}{3}-2r}x^{2r}$, $(-1)^r.\dfrac{2.5.8\ldots(3r-1)}{1.2.3\ldots\ r.3^r}\,a^{-\frac{4}{3}-2r}x^{2r}$.

48. 11046. 49. (i) 0 or $a\left(1 \pm 2\sqrt{\dfrac{b}{c}}\right)$; (ii) $x = \sqrt[3]{\dfrac{b^2c^2}{a}}$, &c.

50. 324 : 105 : 38. 53. 3; $1 + 2y^{-1}$.

54. .2385606, .1003433, .1945378, .0633450, $\bar{1}$.3902320, 1.6136032. 55. 91, 2970. 56. $1+\frac{3}{2}x+\frac{3}{8}x^2-\frac{1}{16}x^3+\frac{3}{128}x^4-$&c.

57. $p + (2m-1)q$. 58. $aR^{p-m} - bR^{p-n}$.

59. $\pm\sqrt{\dfrac{n}{n-2}}$ or $\pm\sqrt{\dfrac{n-1}{n+1}}$. 60. $\frac{1}{216}$, $\frac{125}{216}$, $\frac{75}{216}$.

61. $b(x+y)$. 62. $\frac{9}{4}a^2$, $\frac{1}{4}a^{-4}$. 63. $\dfrac{4m}{4m^2+1}$ a *min.*

64. 12, 84, 4. 65. $\dfrac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, $\dfrac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

66. $-42a^5bc + 105a^4cd^2 - 35a^4b^3 + 210a^3b^2d^2 - 105a^2bd^4 + 7ad^6$;
$21a^5c^2 + 105a^4b^2c - 420a^3bcd^2 + 105a^2cd^4 + 35a^3b^4 - 210a^2b^3d^2 + 105ab^2d^4 - 7bd^6$.

67. $a\left\{\dfrac{n+12}{n-p+q}+1\right\}$ miles. 68. $y = \dfrac{x^2}{a} + \dfrac{x^4}{a^3} + \dfrac{x^6}{a^5} +$ &c.

69. $\pm\dfrac{b}{a}\sqrt{(a^2+b^2)}$. 70. $\frac{1}{2}$, $\frac{1}{2}$. 71. $\dfrac{2(x+1)}{x^2+x+1}$.

74. $(-1)^r.\dfrac{(r+1)\ (r+2)}{1.2}\,x^r$, $\dfrac{1.4.7\ \ldots\ (3r-2)}{1.2.3\ \ldots\ \ r.3^r}\,x^r$,
$-\dfrac{3.1.5.9\ \ldots\ (4r-7)}{1.2.3.4\ \ldots\ \ r.4^r}\,a^{\frac{3}{4}-r}x^r$.

75. $x = a$ gives $\frac{2}{3}$, $x = 2a$ gives $\frac{3}{4}$. 77. $\dfrac{ctt'}{at'+a't}$.

79. $x = \dfrac{a^2(a^2-b^2)}{a^4-b^4}$, $y = \dfrac{a^4-b^4}{a^2(a-b)}$. 80. $4\frac{4}{5}s$.

82. $(ab'-a'b)^2 = (ac'-a'c)^2 + (bc'-b'c)^2$. 83. $\frac{1}{7}$.

84. $\dfrac{3}{2(x-1)} - \dfrac{7}{x-2} + \dfrac{13}{2(x-3)}$, $\dfrac{1}{x^2} + \dfrac{1}{x} + \dfrac{1}{(x-1)^2} - \dfrac{1}{x-1}$.

86. 46.6 ft, 23.3 sq. ft. nearly.

87. $y = b + \frac{1}{3}ab^{-1}x - \frac{1}{81}a^3b^{-5}x^3 + \frac{1}{243}a^4b^{-7}x^4 -$ &c. 88. 2*s* 1*d*.

89. $ab^{\frac{2}{3}} \div (a^{\frac{2}{3}} - b^{\frac{2}{3}})$. 90. $3\frac{3}{8}s$, $5\frac{3}{4}s$, $3\frac{4}{7}s$. 91. 40, 576.

93. $1+\frac{1}{2}ax+(\frac{3}{8}a^2-\frac{1}{2}b)x^2+(\frac{5}{16}a^3-\frac{3}{4}ab)x^3+(\frac{35}{128}a^4-\frac{15}{16}a^2b+\frac{3}{8}b^2)x^4+$&c.

97. $\lfloor n-1\ \{p_0 + p_1 + \text{\&c.}\}\dfrac{r^n-1}{r-1}$; $\frac{1}{2}(r^r-1)\lfloor r$.

98. 1 : 2; 56 : 37. 99. 0, $\dfrac{1}{4}\left\{a-1+\dfrac{1}{a-1}\right\}^2$.

100. $\frac{3}{5}$. 101. $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$. 103. $\frac{a^{3n} - b^{3n}}{b^{3n-3}(a^6 - b^6)}$.

104. $\frac{1}{5}, \frac{5}{13}$; $1, \frac{2}{7}, \frac{1}{10}$. 105. $1\frac{1}{2}$. 106. $20n^2q : m^2p$.

109. (i) $0, \pm\sqrt{(2ab - b^2)}$; (ii) $\frac{1}{2}\{cp \pm \sqrt{(c^2p^2 + 4a^2)}\}$,
where $p = \frac{1}{2}\{b \pm \sqrt{(b^2 - 4)}\}$. 110. £$2\frac{16}{47}$.

111. .6989700, $\bar{2}$.2041200, $\bar{1}$.6989700, .7958800, .2041200, 1.1938200.

113. $\frac{3.8.13\ldots(5r-2)}{1.2.\ 3\ldots\ r}\, a^{-\frac{6}{5}-r}x^r$, $(-1)^r \cdot \frac{5.2.1.4.7\ldots(3r-8)}{1.2.3.4.5\ldots\ r}\, a^{5-r}x^r$,

$(-1)^r \cdot \frac{(r+1)(r+2)(r+3)}{1.2.3}\, a^{-4-r}x^{-4+r}$.

114. $\frac{a}{x^2} + \frac{2}{x} + \frac{2a}{(x-a)^2} - \frac{1}{x-a}$, $\frac{a^2(2x-a)}{3(x^2 - ax + a^2)} + \frac{a^2}{3(x+a)}$.

115. $5(a^4e + 4a^3bd + 2a^3c^2 + 6a^2b^2c + ab^4)$. 117. 5.477225575.

118. $m + n$ or $m + n - 1$; 1 or 0.

119. $x = \frac{1}{30}\{3 \pm 2\sqrt{3} \pm \sqrt{21}\}$; $y = \frac{1}{30}\{3 \mp 2\sqrt{3} \mp \sqrt{21}\}$.

120. A, 4*s*, B, 3*s*; 2*s* $7\frac{1}{2}d$.

121. 12, 48; $x = 2450m^3$, $y = 70m$; $(2n+1)^2$.

123. $\frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{4}{11}$; $5, 5\frac{1}{3}, 5\frac{2}{7}, 5\frac{7}{24}, 5\frac{72}{247}$.

125. $\frac{1}{ax} - \frac{1}{a(x-a)} + \frac{2}{(x-a)^2}$; $\frac{3}{2(x-a)^2} - \frac{1}{2a(x-a)} + \frac{1}{2(x+a)}$

$+ \frac{1}{2a(x+a)}$; $\frac{1}{3a(x-a)} - \frac{1}{ax} + \frac{2(x+2a)}{3a(x^2 + ax + a^2)}$.

126. $\frac{3}{4}, -2, -\frac{7}{16}$. 127. $\frac{mc}{m+1} \cdot \frac{x^2}{a^2}$, $\frac{c}{m+1} \cdot \left(\frac{x}{b}\right)^{\frac{3}{2}}$.

129. (i) $\frac{(b-2c)^2 - ab}{a + 3b - 4c}$; (ii) $\frac{1}{2}[(1 \pm q) \pm \sqrt{\{(1 \pm q)^2 - 4(p \mp r)\}}]$,

where $p = \sqrt[3]{\frac{1-c^2}{2c^4}}$, $q = \sqrt{(1 + 2p)}$, $r = \sqrt{(p^2 + \frac{1}{c^3})}$.

130. 4*d*; 2 to 1. 132. $\sqrt{n} + \sqrt{-n}$.

133. 0, .30103, .4771213, .60206, .69897, .7781513, .845098, .90309, .9542425. 134. $z^3 - 3a^2z + 2b^3 = 0$; $a^7 - b^7 = 7az^2(a^2 - z^2)^2$.

135. $\frac{1}{2}, \frac{3}{7}, \frac{13}{30}, \frac{68}{157}$; $\frac{1}{8}, \frac{3}{25}, \frac{25}{208}, \frac{153}{1273}$; $3, 3\frac{1}{3}, 3\frac{6}{19}, 3\frac{19}{60}$.

138. $\frac{3}{2}r\{2n - 3r + 1\}$.

139. $x = 8, -4, 152 \pm 16\sqrt{6}$; $y = 4, 1, 40 \pm 16\sqrt{6}$.

140. $2\frac{2}{17}d$, 12 to 5; $4\frac{16}{17}d$, 11 to 6. 142. $\frac{2-n}{2^{n+1}}$.

143. $3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}$; $\frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{39}{161}$. 144. $-\frac{p}{2}(1 + \frac{2b}{a})$.

145. $\frac{1}{10}$, $\frac{1}{5}$, $\frac{3}{10}$, $\frac{3}{5}$. 147. $(-1)^n \cdot \dfrac{a-nb+n(n-1)c}{1.2.3\ldots n}$. 148. $\dfrac{Aa}{a+r}$.

149. (i) 0, $\dfrac{4n(1-n^2)}{(1+n^2)^2}$; (ii) $x=0,\ a+b,\ \frac{1}{2}[(a-b)\pm\sqrt{\{(a-b)(a+3b)\}}]$;

$y=0,\ a+b,\ \frac{1}{2}\{(a-b)\mp \&c.\}$.

150. $\frac{13}{24}$, $\frac{143}{238}$, $\frac{3}{32}$. 151. n and $n+1$.

152. $(-2)^{\frac{n}{2}} \cdot \dfrac{1.3.5\ldots(n-1)}{1.2.3\ldots\ \frac{1}{2}n}$, n even; 0, n odd.

153. 19.04872. 155. The 5th in both cases.

159. (i) $x=\sqrt[m+n]{(a^m b^n c^{m+n})}$, $y=\sqrt[m+n]{(a^m b^n)}$; (ii) $\frac{1}{2}(a+b)$.

160. $\frac{4}{13}$, $\frac{5}{14}$. 162. -2. 164. $\frac{1}{3}$, $\frac{1}{3}$.

165. $(-1)^r(r+1)\,a^{-4-2r}x^{2r}$, $\dfrac{1.6.11\ldots(5r-4)}{1.2.\,3\ldots\ r.5^r}\,a^{-\frac{2}{5}-2r}x^{2r}$,

$\dfrac{3.1.1.3.5\ldots(2r-5)}{1.2.3.4.5\ldots\ r.2^r}\,a^{3-2r}x^{2r}$.

169. $x=5,\ 1$; $y=3,\ \frac{3}{5}$. 170. 131 to 112.

171. .50515, .1760913, $\bar{1}$.8239087, 1.1760913, $\bar{3}$.7323939, 1.1583626, .2552726, .9084852. 174. 1, $\frac{81}{20}a^2$.

173. $\dfrac{(n-m+1)a+mc}{n+1}$, $\sqrt[n+1]{(a^{n-m+1}c^m)}$, $\dfrac{(n+1)ac}{ma+(n-m+1)c}$.

175. $\dfrac{1.5.9\ldots(4r-3)}{1.2.3\ldots\ r.4^r}\,a^{-\frac{1}{2}-2r}x^{2r}$, $-\dfrac{1.2.5.8\ldots(3r-4)}{1.2.3.4\ldots\ r.3^r}\,a^{\frac{2}{3}-r}x^r$,

$(-1)^{r-1} \cdot \dfrac{2.3.8\ \ldots(5r-7)}{1.2.3.4\ldots\ r}\,a^{\frac{2}{5}-r}x^r$.

177. $5\frac{3}{5}$ sec. nearly in 24 hours.

178. $n-1$, £nP. 179. $x=3,\ -2$; $y=-2,\ 3$.

180. Chance $=\frac{671}{1296}>\frac{1}{2}$; $\log(1-p)\div\log(\frac{5}{6})$.

182. $r=A \cdot \dfrac{2a-a'}{a^2}$; $n=\dfrac{\log a-\log(a'-a)}{\log(1+r)}$.

183. 3, 4, 5, 6, 7; or -6, $-\frac{1}{2}$, 5, $10\frac{1}{2}$, 16.

184. $\dfrac{(r+1)(r+2)\ldots(r+5)}{1.2\ldots5}\,a^{-2-\frac{r}{3}}x^{-\frac{r}{3}}$, $(-1)^r \cdot c^{-1-\frac{r}{3}}x^{\frac{r}{3}}$.

185. $\dfrac{2}{x}-\dfrac{1}{2(x+1)^2}-\dfrac{5}{4(x+1)}+\dfrac{1}{(x-1)^2}-\dfrac{3}{4(x-1)}$,

$\dfrac{1}{3(x-1)}-\dfrac{1}{x+1}-\dfrac{x-1}{2(x^2-x+1)}+\dfrac{7x-1}{6(x^2+x+1)}$.

186. $\frac{1}{2}(a\sqrt{2}+b)+\frac{1}{2}(a\sqrt{2}+b\sqrt{3})\sqrt{-1}$.

187. $\{pa\pm(p+mq)b\}(a\pm b)^{m-1}$.

189. $ab^{\frac{2}{3}} \div (a^{\frac{2}{3n}} + b^{\frac{2}{3n}})^n$. 190. $\frac{621}{4096}$. 193. $x^{-\frac{1}{3}}$. 194. 1.537.

195. $x + (a - b)x^3 + (a^2 - ab - c + d)x^5 + (a^3 - a^2b - 2ac + ad + bc)x^7 + \{a^4 - a^3b + a^2(b - 2c + d) + a(c - d)\}x^9 + \&c.$

196. $\pm a \dfrac{n^2 + 1}{n^2 - 1}$. 198. $\dfrac{(aq^n - bp^n)r^m + (bp^m - aq^m)r^n}{p^m q^n - p^n q^m}$.

199. $x = 3,\ 2,\ \frac{1}{6}(15 \pm \sqrt{-309})$; $y = \&c.$ 200. $\frac{64}{175}, \frac{46}{175}, \frac{38}{175}, \frac{27}{175}$.

202. $\dfrac{1.3.5 \ldots (2r-1)}{1.2.3 \ldots\ r.2^r} x^{2r}, \ -\dfrac{7.4.1.2.5 \ldots (3r-10)}{1.2.3.4.5 \ldots\ r.3^r} a^{\frac{28}{3} - 2r} x^{2r},$

$\dfrac{1.4.7 \ldots (3r-2)}{1.2.3 \ldots\ r.3^r} a^{-\frac{1}{3} - r} x^r$. 204. $\dfrac{P}{1 - R^{-n}}(R^{-p} - R^{-q})$.

203. .3010380, $\bar{2}$.3010458; 2000.043, 2.000058, .02000082.

205. $3\frac{2}{3}d$. 207. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = z^{\frac{1}{2}}$. 210. $\frac{1}{6}, \frac{2}{3}$. 213. .3053377.

214. $\dfrac{1}{ax^2} + \dfrac{b}{a^2x} - \dfrac{bx + a - a^2c}{a^2(x^2 + a^2)}$. 215. 144 : 235.

216. $am + \dfrac{c}{m}$. 217. $\dfrac{\log(n+2)}{\log 2}$.

219. $\dfrac{1}{24} - \dfrac{(-1)^n}{12(n+1)(n+2)}$; $\dfrac{n}{3(n+1)} - \dfrac{n}{12(n+2)} - \dfrac{n}{18(n+3)}$.

220. $2 - \dfrac{n+2}{2^n}$ shillings. 221. $a^{\frac{mn}{m+n}} + b^{\frac{mn}{m+n}} + c^{\frac{mn}{m+n}} = k^{\frac{mn}{m+n}}$.

222. 50, 50. 223. $1 + 3 + 5 + \&c.$ 225. .99144.

226. $\dfrac{m^4a^2}{(m^2+2)^2}, \dfrac{4m^2a^2}{(m^2+2)^2}, \dfrac{4a^2}{(m^2+2)^2}$; 64+16+1. 227. $\dfrac{\log 2(n+1)}{\log 2}$.

229. $\dfrac{2^{n+1} - 1}{n+1}$, $\frac{1}{12}n(n+1)(n+2)(3n+5)$. 233. $\frac{2}{5}, \frac{3}{7}, \frac{4}{9}$.

234. $\dfrac{1}{2(x+1)^2} + \dfrac{4}{x+1} - \dfrac{8x-7}{2(x^2+1)}$; $\dfrac{1}{(x+1)^2} + \dfrac{8}{x+1} - \dfrac{8x+1}{x^2+x+1}$.

237. $\dfrac{(a-b)^n}{a^{n-1}}$ gallons. 240. $4\frac{1}{2}d$, $7s$ to $1s$.

243. 68, 888, 10456. 244. 1, 3, 5, &c. 245. $6^6 : 22.5^5$.

247. 2, -1, or $\frac{1}{2}(5 \pm \sqrt{17})$. 248. $x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \&c.$

249. $x = 3,\ y = 4$. 250. $23\frac{1}{9}s$ in each case.

253. $\frac{110}{243}a^{-14}b^{12}$, $\frac{874}{1029}$, $1\frac{1}{7}$, $-10830x^4$. 255. $21s$ in each case.

257. $\frac{2}{3}(m^2 - 1)$, $\frac{1}{3}(m^2 + 2)$, $\frac{1}{36}(m^2 - 1)(m^2 - 25)$: 80, 41, 320.

259. The 2nd $= -44\sqrt{2}$. 260. Each $\frac{2}{7}$; $\frac{36}{343}$; $\frac{27}{343}$. 261. 1.

262. $(2x + y - 3)(x - 11y + 1)$. 265. £1 $7s$ $3d$, £1 $2s$.

266. 5455, $\frac{1}{66}\{4.10^x - 15(-1)^x + 11\}$.

267. 360, 519840, 57168810, &c. 268. $\frac{1}{703}$.

269. $(n-1)P\left\{\left(\frac{n}{n-1}\right)^n+(n-1)\right\}$. 270. $\frac{1}{60}$. 271. $\sqrt{\frac{\log p}{\log a}}$.

273. $\frac{1}{6}n(n^2+11)$. 274. 5; $5x+4$. 275. $\frac{1}{2n+3}$.

278. $\frac{1}{12}-\frac{(-1)^n}{4(2n+3)}$; $\frac{1}{12}$. 280. £34 14*s* $5\frac{1}{4}$*d*.

281. 1; $x=a+2c$, $y=b+3c$.

282. $\sqrt{x^m}-\frac{1}{2}\sqrt[np]{(b^p x^2)}$; $(a^2+b^2)(c^2+d^2)$. 284. $\left(\frac{a\pm b}{a}\right)^2$.

286. $\frac{1}{2}$, $\frac{5}{4}$. 287. $\frac{1}{3}\,\frac{(2x)^n-1}{(2x-1)}-\frac{2}{3}\,\frac{(-x)^n-1}{x+1}$.

287. $\frac{1}{3}\cdot\frac{(2x)^n-1}{2x-1}-\frac{2}{3}\cdot\frac{(-x)^n-1}{x+1}$, $\frac{x-1}{(x+1)(2x-1)}$, $\frac{2}{3}\{2^{n-1}+(-1)^n\}$.

288. From the r^{th}, where $r>x$. 290. $\frac{1}{32}$.

292. $x=\frac{1}{2}=y$; $\frac{a(\sqrt{3}-\sqrt{2})+2b}{2a\sqrt{6}+b(\sqrt{2}-\sqrt{3})}$.

293. $a^2+4ac=b^2$. 294. $x=3$, $y=4$.

298. $x=m^2(m^4+1)^2-1$, $y=4m^4(m^4-1)$; if $m=2$, $x=1155$, $y=960$.

302. The 17th. 303. 8 and 9. 304. $\frac{1+4x+x^2}{(1-x)^4}$. 305. $\frac{24}{270725}$.

306. $\frac{\lfloor n\, p^n}{m(m+p)\dots(m+np)}$. 309. $\frac{1}{4}-\frac{1}{4(2n+1)3^n}$, $\frac{1}{4}$.

313. $\frac{p}{\sqrt{(2p^2-1)}}$. 314. $a^3+b^3+c^3+abc=0$. 315. $\frac{53}{140}$, $\frac{13}{28}$.

318. $\frac{1}{2}-\frac{1}{(n+1)(n+2)}$, $\frac{1}{2}$; $\frac{3}{2}-\frac{n+3}{(n+1)(n+2)}$, $\frac{3}{2}$.

319. $x=a^{\frac{n-1}{2}}\{p^n-\frac{n(n-1)}{1.2}p^{n-2}q^2\frac{b}{a}+\&\text{c.}\}$, $y=a^{\frac{n-1}{2}}\{np^{n-1}q-\&\text{c.}\}$.

324. 10 and 6. 325. $\sqrt{2}-1:2\sqrt{3}$. 327. 6, $1\frac{1}{2}$, $\frac{5}{64}$.

328. 30. 329. $\frac{x}{a}+\frac{y}{b}=\sqrt{2}$. 330. $\frac{x^p(x^{q-p+1}-1)}{(q-p+1)(x-1)^2}-\frac{1}{x-1}$.

335. $\frac{n^2(n+1)^2}{2(n^2+1)(n^2+2)}s$. 339. $\frac{n}{12(n+1)}$, $\frac{1}{12}$.

337. $x=4, 11, 20, 31$, &c, $y=9, 11, 13, 15$, &c.

340. $245:243$. 346. $x=\frac{1}{2}\{-1\pm\sqrt{5}\}$ or $\frac{1}{2}\{1\pm\sqrt{-7}\}$. 350. $\frac{1}{16}$.

354. $\frac{ab+ac+bc}{(a+c)(b+c)}$.

EQUATION PAPERS.

1. 1. 35. 2. $x=9$, $y=2$. 3. $3, -\frac{4}{5}$.
4. $x=\pm 12, \pm 9\frac{1}{2}$; $y=2, -1\frac{7}{12}$. 5. 100. 6. 26. 7. 11 : 1.

2. 1. 72. 2. $x=7$, $y=4$. 3. $5, 6\frac{9}{10}$.
4. $x=5, -4\frac{2}{5}$; $y=3, -2\frac{19}{25}$. 5. 180,000.
6. 10. 7. 25 miles.

3. 1. 51. 2. $x=18$, $y=24$. 3. $6, 3\frac{1}{13}$.
4. $x=0, 4$; $y=9, 25$. 5. 22 miles.
6. 18 chains, 30 chains, 21 chains. 7. 10.

4. 1. 3. 2. $x=\frac{4}{3}a$, $y=\frac{5}{4}a$. 3. $4, \frac{1}{9}$. 4. $x=1\frac{1}{4}$; $y=\pm 2$.
5. 10*s*. 6. 79 days. 7. 120 gallons.

5. 1. 7. 2. $x=5$, $y=6$. 3. $4, -\sqrt[3]{49}$.
4. $x=\pm 9, \pm 11\frac{1}{4}$; $y=\pm 4, \pm 3\frac{1}{5}$. 5. 1742.
6. £700, £100. 7. 50 ft, 20 ft, 48 ft.

6. 1. 6. 2. $x=1$, $y=2$, $z=4$. 3. $x=(a^{-\frac{1}{2}}+b^{-\frac{1}{2}})^{\frac{4mn}{m-n}}$.
4. $x=2$ or -1; $y=1$. 5. A mile. 6. £1071 17*s* 6*d*.
7. $\dfrac{c}{1-r}\left\{1+\dfrac{n(n-1)(1-r)^2\{p(1-r)-1\}}{2r\{n(1-r)-(1-r^n)\}}\right\}$.

7. 1. 8. 2. $x=\frac{1}{2}$, $y=\frac{1}{3}$, $z=\frac{1}{4}$. 3. $4, \frac{128}{841}$.
4. $x=2, \frac{1}{18}, \frac{1}{32}$; $y=1, \frac{1}{81}, \frac{1}{36}$. 5. 432.
6. 7 miles, $9\frac{1}{3}$ miles. 7. £96.

8. 1. 4. 2. $x=\pm 4\frac{1}{2}, \pm 3\frac{1}{2}$; $y=\pm 3\frac{1}{2}, \pm 4\frac{1}{2}$.
3. $\left(\dfrac{a+b}{a-b}\right)^{\frac{2pq}{p-q}}$. 4. $x=2, -2, \frac{34}{31}$; $y=2, 6, \frac{1446}{961}$.
5. £60. 6. 60 miles. 7. 4 : 5.

9. 1. $3\frac{3}{8}$. 2. $x=3$, $y=2$. 3. $\left(\dfrac{a^{2b}+1}{a^{2b}-1}\right)^{2}$. 4. $x=5$, $y=3$.
5. £2000. 6. £5 5*s* *per annum*. 7. 60 miles.

10. 1. 11. 2. $x=16$, $y=25$. 3. $3, \frac{3}{16}$.
4. $x=6, 4, 0$; $y=0, 2, -2$. 5. 72, 12.
6. 6 miles per hour. 7. 432.

11. 1. $1\frac{1}{2}$. 2. $x=\pm\sqrt{(ac)}$ or $\frac{1}{2}[(a+c-b)\pm\sqrt{\{(a+c-b)^2-4ac\}}]$;
$y=\pm\sqrt{(bc)}$ or $\frac{1}{2}[(b+c-a)\pm\sqrt{\{(b+c-a)^2-4bc\}}]$.

3. $x = \pm \frac{1}{2}$. 4. $x = \pm \frac{8}{27}$ or $\pm \frac{1}{3}$; $y = \pm \frac{1}{3}$ or $\pm \frac{8}{27}$.
5. 11 o'clock. 6. The number of the tack is the integer equal to or next greater than $\frac{2q}{p} - \frac{2r}{3p}$. 7. 2 hrs.

12. 1. 4. 2. $x=5,\ 1,\ \frac{1}{29}(15\pm6\sqrt{-1})$; $y=3,\ \frac{3}{5},\ \frac{1}{29}(25\pm10\sqrt{-1})$.
3. $\left\{\dfrac{a^{\frac{r}{s}}+1}{a^{\frac{r}{s}}-1}\right\}^{\frac{1}{(m+n)^2}}$. 4. $x=4,\ -2\pm2\sqrt{-3}$; $y=\frac{1}{4},\ -\frac{1}{8}(13\pm3\sqrt{-3})$.
5. 150. 6. 3 miles an hour. 7. £27.

13. 1. 4. 2. $x = 4$ or $\frac{4}{17}$; $y = 12$ or $1\frac{1}{16}\frac{}{}$.
3. 1, 16, or $\frac{1}{2}(1 \pm 3\sqrt{-7})$.
4. $x = 1$ or $1 \pm \sqrt{-2}$; $y = 4$ or -2.
5. 3 miles. 6. £64,000,000. 7. 24 : 21 : 22.

14. 1. $x = 21$; $y = 20$. 2. $\sqrt[3]{(\frac{9}{4}a^2b - b^3)}$.
3, $\pm\sqrt{[\frac{1}{2}\{p \pm \sqrt{(p^2-4)}\}]}$, where $a^2p = -2\{1 \pm \sqrt{(1-a^4)}\}$.
4. $x = \pm\frac{1}{2}\{(4 \pm \sqrt{6}) \pm \sqrt{(18 \pm 8\sqrt{6})}\}$;
$y = \pm\frac{1}{2}\{(2 \pm \sqrt{6}) \pm \sqrt{(18 \pm 8\sqrt{6})}\}$.
5. £20. 6. 2 : 3. 7. 20*O*., 40*S*., 400*P*.

15. 1. $-\frac{1}{2}, -\frac{3}{4}$. 2. $x=(\sqrt{2}+1)^2$ or $(\sqrt{2}-1)^2$; $y=1$ or $(\sqrt{2}-1)^4$.
3. $\frac{1}{2}a$ or $\frac{1}{6}(-5\pm\sqrt{37})a$. 4. $x=4$ or $\frac{1}{16}$; $y=9$ or $\frac{9}{64}$.
5. £240. 6. $54\frac{5}{6}$ miles. 7. *A*, 30; *B*, 163; *C*, 230.

16. 1. 6. 2. $x = 4$; $y = 3$. 3. $9 \pm 4\sqrt{7}$, $\frac{1}{2}(3 \pm \sqrt{13})$.
4. $x=\sqrt{\dfrac{m^2-1}{m^2+1}}$, $y=\sqrt{\dfrac{n^2-1}{n^2+1}}$, where $n^2=\frac{1}{2}\{1\pm\sqrt{3}\pm\sqrt{\pm2\sqrt{3}}\}$,
$m^2=\sqrt{\dfrac{n^4+1}{n^4-1}}$. 5. $4\frac{1}{2}$ A.M. 6. 360 yds. 7. £18, £32.

17. 1. $1\frac{1}{13}$. 2. $x = [b^n\{a^{\frac{1}{2}(m-n)} \pm \sqrt{(a^{m-n} - b^{m-n})}\}]^{\frac{2}{m+n}}$;
$y = [a^m\{b^{\frac{1}{2}(n-m)} \mp \sqrt{(b^{n-m} - a^{n-m})}\}]^{\frac{2}{m+n}}$.
3. $4 \pm \sqrt{6}$, $\pm\sqrt{-2}$. 4. $x = y = \pm a\sqrt{2}$.
5. 1080 yds; $16\frac{1}{2}'$. 6. 8, 12. 7. £30600.

18. 1. 3. 2. $x = 1\frac{1}{2}$, $y = 1\frac{1}{3}$, $z = 1\frac{1}{4}$.
3. -1, $\frac{1}{2}\{p \pm \sqrt{(p^2-4)}\}$, where $p = \dfrac{-(4a+1) \pm \sqrt{\{5(4a+1)\}}}{2(a-1)}$.
4. $x = \dfrac{a-1}{b}$, $y = \{1 + \sqrt{(2a-1)}\}^2$.
5. 6 hrs, 3 hrs. 6. 3 acres. 7. 693, 688, 736.

19. 1. $2\frac{2}{5}$. 2. $x = \pm\frac{3}{2}$ or $\pm\frac{3}{2}\sqrt{-1}$; $y = \pm\frac{1}{2}$ or $\mp\frac{1}{2}\sqrt{-1}$.
3. $x = \frac{2}{3}$, $\pm\frac{4}{3}\sqrt{2}$, $\frac{4}{9}(-2 \pm \sqrt{-14})$.
4. $x=0$ or 1, $y = \pm\sqrt{a}$; or $x = \frac{1}{2}\{1 \pm \sqrt{(1+4p)}\}$, $y = \pm\sqrt{\left(\frac{a}{b}(b-p)\right)}$,

$$\text{where } p = \frac{\{2a^2 - (a-1)\,b\}\,b}{a^2 - ab + b^2}.$$

5. $10\frac{1}{2}$ minutes. 6. 2*s* 1*d*. 7. 3300 gals, 1800 gals.

20. 1. 3. 2. $x = 1 + a$; $y = \frac{1}{2}\{\sqrt{(1 - a + a^2)} + \frac{1}{2} - a\}$;
$z = \frac{1}{2}\{\sqrt{(1 - a + a^2)} - \frac{1}{2} + a\}$.
3. $4\sqrt{2}$ 4. $x = \frac{1}{2}\{\sqrt{(\sqrt[3]{3} + 3)} \pm \sqrt{(\sqrt[3]{3} - 1)}\}$;
$y = \frac{1}{2}\{\sqrt[3]{3}\sqrt{(\sqrt[3]{3} + 3)} \pm \sqrt{(3\sqrt[3]{9} - 1)}\}$.
5. Money £672, debts £840. 6. 15′, 7. 10.

21. 1. $-2, \frac{1}{3}$. 2. $x = \pm 1, 2, -2\sqrt[3]{4}$; $y = \pm 1, 2, -\sqrt[3]{2}$; $z = 4, 1, 1$.
3. $1, -3, -\frac{1}{2}$. 4. $x = \pm\sqrt{(\frac{8}{13} + \frac{2}{13}\sqrt{3})}$; $y = \pm\sqrt{(\frac{11}{13} + \frac{6}{13}\sqrt{3})}$.
5. 1600. 6. 6. 7. 16 miles.

22. 1. ± 1 or ± 8. 2. $x = 6$ or $3\frac{1}{3}$, $y = 5$ or $6\frac{1}{3}$, $z = 3$ or $4\frac{1}{3}$.
3. $\pm\frac{a}{2}\{\sqrt{\frac{1+2n}{n}} \pm \sqrt{\frac{1-2n}{n}}\}$ or $\pm\frac{a}{2}\{\sqrt{\frac{5n-3}{n-1}} \pm \sqrt{\frac{n+1}{n-1}}\}$.
4. $x^3 = (a^2 - 2bc - 2b^2)\,\{c \pm \sqrt{(c^2 - a^2 + 2bc + 2b^2)}\}$;
$y^3 = (a^2 - 2bc - 2b^2)\,\{c \mp \&c.\}$.
5. 195. 6. 11, 18500, 1789. 7. 12.

23. 1. $\frac{65}{66}$. 2. $x = 3$, $y = -2\frac{1}{2}$. 3. $3, -\frac{1}{3}, \frac{1}{3}(4 \pm \sqrt{-2})$.
4. $x = \pm\frac{1}{2}\sqrt{5}$ or 0, $y = \pm\sqrt{\frac{3}{2}}$ or $\pm\sqrt{-1}$;
$x = \pm\frac{1}{3}\sqrt{\{\frac{1}{2}(1 \pm \sqrt{33})\}}$, $y = \pm\frac{1}{2}\sqrt{(5 \pm \sqrt{33})}$.
5. 84 miles. 6. $1\frac{1}{2}$ in., $\frac{1}{4}$ in. 7. £180; 44 : 35.

24. 1. 2 or 3. 2. 49, 64, $\frac{1}{2}(93 \pm \sqrt{185})$.
3. $\pm 1\frac{1}{6}$, $\pm 1\frac{1}{14}$, $-1\frac{1}{2}$, $\pm\frac{3}{2}\sqrt{-1}$.
4. $x = \pm\frac{1}{2}a$ or $\pm a\sqrt{-2}$; $y = \pm\frac{1}{2}a$ or $\mp a\sqrt{-\frac{1}{2}}$.
5. £400. 6. 41 cub. in. 7. 20 miles.

25. 1. 3. 2. $\pm\sqrt{\{a^2 - (a-b)^2\}}$ or $\pm\sqrt{\{a^2 - (a+3b)^2\}}$.
3. $x = \frac{7}{2}(3 \pm \sqrt{33})$; $y = \frac{7}{2}(3 \mp \sqrt{33})$.
4. $x = \frac{a}{2}(1 - \sqrt{3})$; $y = \frac{a}{2}(1 - \frac{1}{\sqrt{3}})\sqrt{(1 - \frac{4}{\sqrt{3}})}$. 5. $31\frac{1}{2}$.
6. $\frac{m'}{(a-a') + n(c'-c)}$, $\frac{m'n}{(a-a') + n(c'-c)}$. 7. 100.

APPENDIX.

For the following proofs I am indebted to the kindness of the Rev. J. Griffith, M.A., of St. John's College.

I. *Binomial Theorem.*

Let $P_r = \dfrac{p(p-1)\ldots(p-r+1)}{1.2\ldots r}, \quad Q_r = \dfrac{q(q-1)\ldots(q-r+1)}{1.2\ldots r}.$

whatever p and q may be: then

$$rP_r = (p-r+1)P_{r-1}, \quad (r-1)P_{r-1} = (p-r+2)P_{r-2}, \quad \&c.,$$
$$2P_2 = (p-1)P_1, \quad P_1 = p;$$

and $rQ_r = (q-r+1)Q_{r-1}, \quad (r-1)Q_{r-1} = (q-r+2)Q_{r-2}, \quad \&c.,$

$$2Q_2 = (q-1)Q_1, \quad Q_1 = q.$$

Now, by actual Multiplication, the product of the two series,

$(1+P_1x+P_2x^2+\&c.)\times(1+Q_1x+Q_2x+\&c. = 1+C_1x+C_2x^2+\&c.$ suppose,

where $C_1 = P_1 + Q_1 = p+q,$

$$C_{r-1} = P_{r-1} + P_{r-2}Q_1 + P_{r-3}Q_2 + \&c. + P_1Q_{r-2} + Q_{r-1},$$
$$C_r = P_r + P_{r-1}Q_1 + P_{r-2}Q_2 + \&c. + P_1Q_{r-1} + Q_r;$$

therefore rC_r

$$\begin{aligned}
&= rP_r + \{1+(r-1)\}P_{r-1}Q_1 + \{2+(r-2)\}P_{r-2}Q_2 + \&c. + rQ_r \\
&= rP_r + (r-1)P_{r-1}Q_1 + (r-2)P_{r-2}Q_2 + \&c. + P_1Q_{r-1} \\
&\quad + P_{r-1}Q_1 + 2P_{r-2}Q_2 + 3P_{r-3}Q_3 + \&c. + rQ_r \\
&= (p-r+1)P_{r-1} + (p-r+2)P_{r-2}Q_1 + (p-r+3)P_{r-3}Q_2 + \&c. + pQ_{r-1} \\
&\quad + qP_{r-1} + (q-1)P_{r-2}Q_1 + (q-2)P_{r-3}Q_2 + \&c. + (q-r+1)Q_{r-1} \\
&= (p+q-r+1)P_{r-1} + (p+q-r+1)P_{r-2}Q_1 + (p+q-r+1)P_{r-3}Q_2 + \&c. + (p+q-r+1)Q_{r-1} \\
&= (p+q-r+1)\{P_{r-1} + P_{r-2}Q_1 + P_{r-3}Q_2 + \&c. + Q_{r-1}\} = (p+q-r+1)C_{r-1}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
C_1 &= (p+q) \\
2C_2 &= (p+q-1)C_1 \\
3C_3 &= (p+q-2)C_2 \\
\&c. &= \&c. \\
rC_r &= (p+q-r+1)C_{r-1};
\end{aligned}$$

$\therefore$, multiplying, and cancelling,

$$1.2.3\ldots rC_r = (p+q)(p+q-1)\ldots(p+q-r+1);$$

and the product required is, consequently,

$$1 + (p+q)x + \frac{(p+q)(p+q-1)}{1.2}x^2 + \&c.$$
$$+ \frac{(p+q)(p+q-1)\ldots(p+q-r+1)}{1.2\ldots r}x^r + \&c.;$$

that is, the product of two such series is exactly a similar series, with the sum of the *characteristics* (p and q) in place of either.

Hence the continued product of three such series, whose characteristics are p, q, r, is the product of two similar series, whose characteristics are $p+q$ and r, which is also a similar series with characteristic $p+q+r$. And so on for the product of any number of such series. So that, generally, the product of k such series is an exactly similar series whose characteristic is the sum of their k characteristics; and, therefore, the k^{th} power of such a series is a similar series, with its characteristic k times as great.

Hence, if h be a positive integer, we have

$$(1+x)^h \text{ or } \{1 + 1x + \frac{1(1-1)}{1.2}x^2 + \frac{1(1-1)(1-2)}{1.2.3}x^3 + \&c.\}^h$$
$$= 1 + hx + \frac{h(h-1)}{1.2}x^2 + \frac{h(h-1)(h-2)}{1.2.3}x^3 + \&c.,$$

which proves the theorem for a positive integral index.

Again, if h and k be positive integers, we have

$$\{1 + \frac{h}{k}x + \frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1.2}x^2 + \&c.\}^k = 1 + hx + \frac{h(h-1)}{1.2}x^2 + \&c. = (1+x)^h;$$
$$\therefore (1+x)^{\frac{h}{k}} = 1 + \frac{h}{k}x + \frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1.2}x^2 + \&c.,$$

which proves the theorem for a positive fractional index.

Lastly, by the previous reasoning of this Article, we have

$$\{1 + hx + \frac{h(h-1)}{1.2}x^2 + \&c.\}\{1 + (-h)x + \frac{-h(-h-1)}{1.2}x^2 + \&c.\}$$
$$= 1 + (h-h)x + \frac{(h-h)(h-h-1)}{1.2}x^2 + \&c. = 1;$$
$$\therefore 1 + (-h)x + \frac{-h(-h-1)}{1.2}x^2 + \&c. = \frac{1}{1+hx+\&c.} = \frac{1}{(1+x)^h} = (1+x)^{-h},$$

which proves the theorem for a negative index.

COR. Hence, therefore,

$$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n = a^n\{1 + n\left(\frac{x}{a}\right) + \frac{n(n-1)}{1.2}\left(\frac{x}{a}\right)^2 + \&c.\}$$
$$= a^n + na^{n-1}x + \tfrac{1}{2}n(n-1)a^{n-2}x^2 + \&c.,$$

where coefficient of

$$a^{n-r}x^r = \frac{n(n-1)\dots(n-r+1)}{1.2\dots r} =, \text{ as in (182), } \frac{\lfloor n}{\lfloor r\,\lfloor n-r}.$$

II. *Exponential Theorem.*

By actual multiplication, we have

$$\left\{1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \&c. + \frac{x^r}{\lfloor r} + \&c.\right\} \times \left\{1 + \frac{y}{\lfloor 1} + \frac{y^2}{\lfloor 2} + \&c. + \frac{y^r}{\lfloor r} + \&c.\right\}$$

$$= 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \&c. + \frac{x^r}{\lfloor r} + \&c.$$
$$+ \frac{y}{\lfloor 1} + \frac{y}{\lfloor 1} \cdot \frac{x}{\lfloor 1} + \&c. + \frac{y}{\lfloor 1} \cdot \frac{x^{r-1}}{\lfloor r-1} + \&c.$$
$$+ \frac{y^2}{\lfloor 2} + \&c. + \frac{y^2}{\lfloor 2} \cdot \frac{x^{r-2}}{\lfloor r-2} + \&c.$$
$$+ \&c. + \&c. \qquad + \&c.$$
$$= 1 + \frac{x+y}{\lfloor 1} + \frac{(x+y)^2}{\lfloor 2} + \&c. + \frac{(x+y)^r}{\lfloor r} + \&c.;$$

that is, the product of two such series is an exactly similar series with $x+y$ in the place of x or y. So the product of k such series is an exactly similar series with the sum of their k characteristics in place of either of them; and the k^{th} power of such a series is an exactly similar series with its characteristic k times as great.

Hence, if $x = \frac{h}{k}$, where h and k are both positive integers, we have

$$\left\{1 + \frac{x}{1} + \frac{x^2}{1.2} + \&c.\right\}^k = 1 + \frac{kx}{1} + \frac{(kx)^2}{1.2} + \&c. = 1 + \frac{h}{1} + \frac{h^2}{1.2} + \&c.$$
$$= \left\{1 + \frac{1}{1} + \frac{1^2}{1.2} + \&c.\right\}^h = e^h, \text{ suppose};$$

$\therefore\ 1 + \frac{x}{1} + \frac{x^2}{1.2} + \&c. = e^{\frac{h}{k}} = e^x$, where x must be a *positive* fraction or integer, in which latter case $k = 1$.

Again, if $x = -m$, a *negative* fraction or integer, then

$$\left\{1 + \frac{x}{1} + \frac{x^2}{1.2} + \&c.\right\} \times \left\{1 + \frac{m}{1} + \frac{m^2}{1.2} + \&c.\right\}$$
$$= 1 + \frac{x+m}{1} + \frac{(x+m)^2}{1.2} + \&c. = 1, \text{ since } x + m = 0;$$
$$\text{or } 1 + \frac{x}{1} + \frac{x^2}{1.2} + \&c. = \frac{1}{1 + \frac{m}{1} + \frac{m^2}{1.2} + \&c.} = \frac{1}{e^m} = e^{-m} = e^x.$$

So that, whether x be integral or fractional, positive or negative,

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.; \text{ and } a^x = (e^{\log_e a})^x = e^{x \log_e a};$$
$$\therefore\ a^x = 1 + \frac{x \log_e a}{1} + \frac{(x \log_e a)^2}{1.2} + \frac{(x \log_e a)^3}{1.2.3} + \&c.$$

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